Complex Analysis Qualifying Examination

January 1996

1. Find the Laurent series for the function

$$f(z) = \frac{4z - z^2}{(z^2-4)(z+1)}$$

valid in the annulus $\{1 < |z| < 2\}$.

2. Find a one-to-one conformal mapping from the intersection of the disks $|z| < 1$ and $|z-1| < 1$ onto the unit disk $|z| < 1$.

3. Evaluate the definite integral

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx.$$

4. Fix $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the right half-plane $\{\operatorname{Re} z > 0\}$. In addition, show that this solution must be real.

5. Let G be a bounded, connected, open set in the complex plane. Suppose that f and g are continuous functions on the closure of G with no zeros, and f and g are holomorphic in G. Show that if $|f(z)| = |g(z)|$ for all $z \in \partial G$, then $f = \lambda g$ for some constant λ with $|\lambda| = 1$.

6. Consider the family $\mathcal{F}$ of analytic functions f defined on the unit disk D such that $f(0) = i$ and $\operatorname{Im} f(z) \geq 0$ for all z in D. Find $\sup\{\,|f'(0)| : f \in \mathcal{F}\,\}$.

7. State the Riemann mapping theorem, the Runge approximation theorem, and the great Picard theorem. Sketch the proof of one of these three theorems.

8. Determine all real-valued harmonic functions u defined in the annulus $\{\, z \in \mathbf{C} : 1 < |z| < 2\,\}$ that are independent of the argument of z. (In other words, $u(z) = u(|z|)$ for every z in the annulus.) Prove that you have found them all.

9. Let f be an entire function normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Find necessary and sufficient conditions on f for normality of the family of successive iterates $\{f \circ f, f \circ f \circ f, f \circ f \circ f \circ f, \ldots\}$. (Here $\circ$ denotes function composition.) Does it make a difference whether normality is interpreted with respect to the Euclidean metric or with respect to the spherical metric?

10. Determine all holomorphic automorphisms of the annulus $\{\, z \in \mathbf{C} : 1 < |z| < 2\,\}$. In other words, find all one-to-one analytic functions that map the annulus onto itself. Prove that you have found them all. You may assume that automorphisms of the annulus automatically extend continuously to the boundary; extra credit if you can solve the problem without making this assumption.