Complex Analysis Qualifying Examination

January 13, 1999

1. Determine the coefficient a_{1999} in the Laurent series

$$\frac{z}{(z-1)(z-13)} = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad 1 < |z| < 13.$$

2. Use contour integration to evaluate the real integral $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$, where $0 < a < 1$.

3. Construct a holomorphic function whose domain is the unit disk $\{\, z \in \mathbb{C} : |z| < 1 \,\}$ and whose image is the punctured disk $\{\, z \in \mathbb{C} : 0 < |z| < 1 \,\}$.

4. If u is a continuous, real-valued, subharmonic function on an open set in the plane, when is u^2 also subharmonic? (Always? Never? Sometimes?) Support your conclusions with appropriate proof or examples.

5. Suppose that f is a continuous function on the closed square $\{\, z \in \mathbb{C} : |\operatorname{Re} z| \le 1$ and $|\operatorname{Im} z| \le 1 \,\}$, and f is holomorphic in the interior of the square, and f is the identity function on the boundary of the square. Must f be the identity function in the interior of the square? Supply a proof or a counterexample, as appropriate.

6. Determine a rectangle inside which there is exactly one solution of the equation $\cos(z) = iz$.

7. State and prove a version of the Schwarz reflection principle.

8. Show that there exists an entire function f such that $f(n) = f'(n)$ for every integer n, and such that the range of f includes both 0 and 1.

9. Prove the following particular case of the maximum principle with an exceptional point: If f is holomorphic in the unit disk $\{\, z \in \mathbb{C} : |z| < 1 \,\}$; and $|f(z)| < 1$ when $|z| < 1$; and $\limsup_{z \to w} |f(z)| \le 1/2$ for every point w such that $|w| = 1$, with the possible exception of the point $w = 1$; then $|f(z)| \le 1/2$ when $|z| < 1$.

10. For *one* of the following, state the theorem and sketch its proof: Runge's approximation theorem; Mittag-Leffler's theorem on meromorphic functions with prescribed poles; Picard's theorem on the range of entire functions.