Complex Analysis Qualifying Examination

May 2001

All problems are worth 10 points.

1) Show: Four points z_1, z_2, z_3, z_4 in $\mathbb{C}$ lie on a straight line or on a circle if and only if their cross ratio (z_1, z_2, z_3, z_4) is real.

2) Define $\mathcal{F}$ to be the family of holomorphic functions which map the open unit disc to itself and which together with their first three derivatives vanish at 0. Find $\sup_{f \in \mathcal{F}} |f(1/2)|$.

3) Let f be continuous on $\mathbb{C}$ and holomorphic on the complement of the closed rays $\{re^{i\pi/n} | r \geq 0\}, n = 1, 2, 3, \ldots$. Show that f is an entire function (i.e. f is actually holomorphic on all of $\mathbb{C}$).

4) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on the open unit disc D such that for all $z \in D, \{|f_n|\}_{n=1}^{\infty}$ is uniformly bounded in a neighborhood of z. Assume that $\lim_{n\to\infty} f_n(1/k) = 0, k = 2, 3, 4, \ldots$. Show that $f_n(z) \to 0$ uniformly on compact subsets of D.

5) Suppose $A \subseteq \mathbb{C}$ is a closed, connected set and that $\mathbb{C} \setminus A$ is connected. Suppose f is entire with an even number of zeros (counting multiplicities) in A and that f never vanishes on $\mathbb{C} \setminus A$. Show that a holomorphic branch of the square root of f exists on $\mathbb{C} \setminus A$.

6) Let $p(z)$ and $q(z)$ be holomorphic polynomials, and let n be the degree of p. Show:

(a) The set Ω of those $z \in \mathbb{C}$ where $p^{-1}(z)$ consists of n points is open.

(b) $\tilde{q}(z) = \sum_{p(w)=z} q(w)$ is holomorphic on Ω.

7) Assume f is continuous on the domain Ω and satisfies the mean value property for circles (i.e. the value at the center is the mean of the values over the circle). Show that f is harmonic.

8) Write down an entire function f which has simple zeros at $z = 1, 2, 3, \ldots$ and no other zeros; your answer should be given as an infinite product $f(z) = \prod g_n(z)$, with an explicit formula for $g_n(z)$. Does f have a removable singularity, a pole, or an essential singularity at ∞? Justify your answers.

9) For a compact subset K of $\mathbb{C}$, define $\widehat{K}$ to be the set of those $z \in \mathbb{C}$ where $|p(z)| \leq \max_{\zeta \in K} |p(\zeta)|$ for all holomorphic polynomials p. Show that $\widehat{K}$ is the union of K and all the bounded components of $\mathbb{C} \setminus K$.

10) Carefully state (but do not prove) the following theorems:
(a) the Residue Theorem.
(b) the Monodromy Theorem.
(c) the Riemann Mapping Theorem.