

Complex Analysis Qualifying Examination
May 2002

All problems are worth 10 points.

- 1) Use contour integration to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2} dx$$

when a is an arbitrary real number.

- 2) Consider the infinite product

$$\prod_{n=0}^{\infty} (1 + z^{2^n}) .$$

- a) Prove that this product converges for $|z| < 1$ and diverges for $|z| > 1$.
b) Show that when $|z| < 1$, the product equals $1/(1-z)$.

- 3) Find

$$\int_{\gamma} \frac{z}{(z+3i)(z+2)^2} dz ,$$

where γ is the curve in the figure below.

4) Let K be a compact subset of \mathbb{C} whose complement contains at least one bounded component. Let z_0 be a point in a bounded component of $\mathbb{C} \setminus K$. Show that the function $1/(z - z_0)$ cannot be approximated uniformly on K by polynomials.

5) Suppose f is meromorphic in a neighborhood of the closed unit disc \overline{D} , with only a simple pole at a point $z_0 \in D$. If $f(\partial D) \subseteq \mathbb{R}$, show that there are constants $a \in \mathbb{C}$ and $b \in \mathbb{R}$ such that

$$f(z) = \frac{az^2 + bz + \bar{a}}{(z - z_0)(1 - \bar{z}_0 z)}.$$

6) Let \mathcal{F} be a family of functions $f(z) = \sum_{n=0}^{\infty} a_n(f)z^n$ that are holomorphic on the unit disc. Define $A_n = \sup_{f \in \mathcal{F}} |a_n(f)|$. Show that \mathcal{F} is a normal family if and only if A_n is finite for all n and $\limsup_{n \rightarrow \infty} (A_n)^{1/n} \leq 1$.

7) Show that there is no function $f(z)$ meromorphic in a neighborhood of zero which satisfies

$$z^2 f'(z) = (f(z))^2 + z.$$

8) Consider the Monodromy Theorem.

a) State the theorem.

b) *Sketch* a proof of the theorem. (You only need to indicate the main steps!)

9) Suppose Ω is a simply connected domain in \mathbb{C} , f is holomorphic in Ω , and the closure of the image $f(\Omega)$ is a compact subset of Ω . Prove that f has a fixed point, that is, there is a point $z_0 \in \Omega$ such that $f(z_0) = z_0$.

10) Carefully state (but do not prove) the following theorems:

(a) Runge's Theorem.

(b) The Mittag-Leffler Theorem.

(c) Harnack's Principle.