Complex Analysis Qualifying Exam, May 2003.

Each problem is worth 10 points.

1) Let f be an entire function. Denote $M(r) = \max_{|z|=r} |f(z)|$. Suppose that $M(2r)/M(r)$ has a finite limit as $r \to \infty$. Prove that f is a polynomial.

2) Let f be analytic in the unit disk $\mathbb{D}$ and continuous up to the boundary. Suppose that 0 lies in the image $f(\mathbb{D})$ and that $|f| > 0$ on the unit circle $\mathbb{T}$. Prove that there exists a function g analytic in $\mathbb{D}$ and continuous up to the boundary such that $|f| = |g|$ on $\mathbb{T}$ and $|g| > |f|$ for any $z \in \mathbb{D}$.

3) Let f be a holomorphic function in a complex domain Ω. Suppose that $|f|^2 = p(x, y)$ in Ω, where $p(x, y)$ is a polynomial of two variables. Prove that then f is a polynomial of z. (Hint: apply $(\frac{\partial}{\partial z})^n$.)

4) Let f be an entire function. Prove that for every $C > 0$ there exists $z, |z| = C$ such that $|zf(z) - e^z| \geq \exp(Rez)$.

5) Let v be a function of two variables defined in $\Omega = \{|z - 1| < 1\}$

$$v(x, y) = (x^2 - y^2)\arg(x + iy) + xy\ln(x^2 + y^2).$$

Does there exist a function u such that $f = u + iv$ is analytic in Ω? If the answer is negative give the proof, otherwise produce f.

6) State Mergelyan's Theorem (approximation by polynomials) and the Big Picard Theorem (essential singularities).

7) Let f and g be two holomorphic functions in the unit disk, both are 1-1, $g(\mathbb{D}) \subset f(\mathbb{D})$, $g(0) = f(0)$. Prove that for any $0 < r < 1$, $g(\{|z| < r\}) \subset f(\{|z| < r\})$.

8) Let $F = \{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions in $\mathbb{D}$ satisfying

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < C < \infty$$

for each n (i. e. all f_n belong to H^2 with $||f_n||_{H^2}^2 < C < \infty$ for any n). Prove that F is a normal family.

9) Calculate $\oint_{\{|z|=R\}} \frac{z}{1-e^z} dz$ where $R > 0, R \neq 2\pi n$.

10) Let f be holomorphic in a neighborhood of 0 and suppose that it satisfies $f(2z) = 2f(z)f'(z)$ in that neighborhood. Prove that f can be extended to an entire function.