Complex Analysis Qualifying Examination

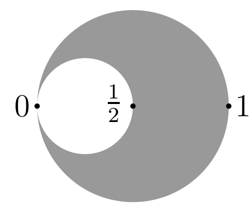
May 2004

1. Give an example of a simply connected domain G in $\mathbb{C}$ and a holomorphic function f defined on G such that the image $f(G)$ is not simply connected and the derivative f' has no zeroes.

2. Prove that if f is a periodic entire function with period $2\pi i$ (in other words, $f(z) = f(z + 2\pi i)$ for all z), then there exists a holomorphic function g on the domain $\mathbb{C} \setminus \{0\}$ such that $f(z) = g(e^z)$ for all z.

3. Use contour integration to prove that the Fresnel integral $\int_0^\infty \cos(x^2)\, dx$, viewed as an improper Riemann integral, converges to the value $\sqrt{2\pi}/4$. Hint: $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}/2$.

4. Find an explicit biholomorphic mapping from $\{\, z \in \mathbb{C} : |z - \frac{1}{2}| < \frac{1}{2} \text{ and } |z - \frac{1}{4}| > \frac{1}{4} \,\}$ (the region illustrated in the figure) to the unit disc $\{\, z \in \mathbb{C} : |z| < 1 \,\}$.



5. Suppose that the Laurent series $\sum_{k=-\infty}^{\infty} a_k z^k$ converges to $1/\sin(\pi z)$ when $0 < |z| < 1$, and suppose that the Laurent series $\sum_{k=-\infty}^{\infty} b_k z^k$ converges to $1/\sin(\pi z)$ when $1 < |z| < 2$. Find, for every integer k, a simple expression for the difference $(b_k - a_k)$.

6. Let f be a holomorphic function in a neighborhood of the closed unit disc $\{\, z \in \mathbb{C} : |z| \leq 1 \,\}$, and suppose that $\operatorname{Re}(\overline{z} f(z)) > 0$ when $|z| = 1$. Prove that f has exactly one zero in the open unit disc.

7. Let f be an entire function. Prove that if f takes purely real values on some pair of lines intersecting at an angle of $\pi/\sqrt{2}$ radians, then f is a constant function.

8. Suppose that a sequence $\{f_n\}_{n=1}^{\infty}$ of holomorphic functions is defined recursively as follows: $f_1(z) = \sin z$, and $f_n(z) = f_{n-1}(\sin z)$ for $n \geq 2$. Is this sequence of functions a normal family in the open unit disc $\{\, z \in \mathbb{C} : |z| < 1 \,\}$? Prove your answer.

9. Let f be an entire function such that $f(n) = 0$ for every integer n. Prove that if $|f(z)| \leq 2004 e^{|z|}$ for all z, then f is identically equal to 0.

10. For *one* of the following, state the theorem and sketch its proof: the Riemann mapping theorem, Picard's little theorem, Mittag-Leffler's theorem.