Complex Analysis Qualifying Examination

May 2005

All problems are worth 10 points.

1. Let u be a harmonic function in a domain $\Omega \subseteq \mathbb{C}$. Suppose that the set $\{ z \in \Omega \mid \frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0 \}$ has a limit point in Ω. Show that u is a constant function.

2. Let $p(z)$ be a polynomial of degree n, where $n \geq 1$, and let $a > 0$. What is the maximal number of connected components that the set $\{ z \in \mathbb{C} \mid |p(z)| < a \}$ can have? Prove your assertion.

3. Either find, or show that there does not exist, a holomorphic function f in the unit disc such that $f(1/n) = f(-1/n) = 1/(2n+1)$ for every integer $n \geq 2$.

4. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}$, and $\lim_{n\to\infty} f_n(z_0) = 0$ for some $z_0 \in \Omega$. Assume additionally that the sequence $\{\mathrm{Re}(f_n)\}_{n=1}^{\infty}$ converges to 0 locally uniformly in Ω. Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges to 0 locally uniformly in Ω.

5. A compact set $K \subset \mathbb{C}$ is called polynomially convex if, whenever $z_0 \in \mathbb{C} \setminus K$, there exists a polynomial $p(z)$ such that $\max_{z\in K} |p(z)| < |p(z_0)|$. Show that a compact set K is polynomially convex if and only if $\mathbb{C} \setminus K$ is connected.

6. State and prove Jensen's formula.

7. Use residue calculus to prove that $\displaystyle\int_0^{\infty} \frac{\sqrt{x}}{x^4+1}\,dx = \frac{\pi\cos(\pi/8)}{2+\sqrt{2}}$.

8. Let $\mathcal{F}$ be the family of all *injective* holomorphic functions f defined on the unit disc such that $f(0) = 0$ and $f'(0) = 1$. Show that $\mathcal{F}$ is closed in the topology of locally uniform convergence.

9. Let f be an entire function that is real-valued on the real axis and that satisfies $\mathrm{Im}(f(z)) > 0$ when $\mathrm{Im}(z) > 0$. Show that there exist a real number b and a positive real number a such that $f(z) = az + b$ for all z.

10. Prove that if f and g are entire functions such that $f(z)^2 + g(z)^2 = 1$ for all z, then there exists an entire function h such that $f(z) = \cos(h(z))$ and $g(z) = \sin(h(z))$ for all z.