Complex Analysis Qualifying Examination

May 2006

1. Determine the largest open set of points z in the complex plane for which the series $\sum_{n=1}^{\infty} \frac{\cos(nz)}{2^n}$ converges.

2. Suppose $f(z) = \dfrac{z}{(1-z)^3}$. Show that f is a one-to-one function on the open disc centered at 0 with radius $\frac{1}{2}$ but on no larger open disc centered at 0.

3. Prove that the integral $\displaystyle\int_0^\infty \frac{e^{-t} - e^{-zt}}{t}\, dt$ converges when $\operatorname{Re} z > 0$ and represents the principal branch of $\log(z)$ in that region.

4. Suppose f is a holomorphic function on the punctured complex plane $\mathbb{C} \setminus \{0\}$ with the property that $f\left(\dfrac{1}{z}\right) = f(z)$ for every nonzero complex number z. Show that there exists an entire function g with the property that $f(z) = g\left(z + \dfrac{1}{z}\right)$ for every nonzero complex number z.

5. Prove that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\, dx = \frac{\pi}{e}$.

6. Determine the number of points z in the open unit disc for which $e^z = 3z^4$.

7. Determine the number of points z in the complex plane for which $e^z = 3z^4$.

8. Suppose a holomorphic function in a neighborhood of the point i admits analytic continuation along two different paths to a neighborhood of 0. If the Maclaurin series of the two continuations are different, must the radii of convergence of these Maclaurin series be less than or equal to 1? Supply a proof or a counterexample, as appropriate.

9. Consider the family of holomorphic functions f on the unit disc such that $\operatorname{Re} f(z) \leq \operatorname{Im} f(z)$ for all z, and $f(0) = -1$. Is this family a normal family? Explain.

10. State **one** of the following theorems and sketch its proof: the Riemann mapping theorem, Mittag-Leffler's theorem about meromorphic functions with prescribed singularities, or Runge's approximation theorem.