Complex Analysis
Qualifying Examination

May 2007

Each problem is worth 10 points.

1. Let $\sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n. Prove that if $|a_0| < |a_n|$, then the polynomial has at least one zero inside the unit disk $\{\, z \in \mathbb{C} : |z| < 1 \,\}$.

2. Suppose that u is a non-constant real-valued harmonic function on a connected open subset of the plane, and let p be a real-valued polynomial of one real variable. Prove that if the composite function $p \circ u$ is a harmonic function, then the degree of p is less than 2.

3. Evaluate $\dfrac{1}{2\pi i} \displaystyle\int_C \frac{z^{2007}}{2z^4 + 1}\, dz$, where C is the unit circle with its standard counter-clockwise orientation.

4. Determine the holomorphic automorphism group (that is, the set of holomorphic bijections) of the twice-punctured plane $\mathbb{C} \setminus \{0, 1\}$.

5. Prove that $\sqrt{\dfrac{z+1}{z-1}}$ has a holomorphic branch in the region where $|z| > 1$, and show that the coefficient of $1/z^3$ in the Laurent series expansion equals $\pm 1/2$.

6. Let $f(z) = z + e^z$. Determine the image under f of the horizontal strip where $0 < \operatorname{Im} z < \pi$.

7. Suppose f is an entire function such that $\sin(\cos(f(z))) = \cos(\sin(f(z)))$ for all z. Prove that f is a constant function.

8. Suppose f is a holomorphic function that maps the disk $\{\, z \in \mathbb{C} : |z| < 4 \,\}$ into itself. Prove that if $f(1) = 2$, then $|f'(1)| \le 4/5$.

9. Suppose f is a holomorphic function in the unit disk, and let $\mathcal{F}$ denote the family of derivatives of f of all orders. Prove that if $\mathcal{F}$ is a normal family (in the sense that every sequence of elements admits a subsequence converging uniformly on compact sets to a holomorphic function), then f is the restriction to the unit disk of an entire function.

10. Prove that the infinite product $\prod_{n=1}^{\infty} \cos(\frac{z}{n})$ converges for all z and represents an entire function of order 1.