# Complex Analysis Qualifying Examination

26 May 1995

1. Let $S = \{ z \in \mathbb{C} : e^{e^z} = 1 \}$. Find the distance from the set $S$ to the point $i$, that is, find $\inf_{z \in S} |z - i|$.

2. Evaluate $\displaystyle\int_{|z|=3} \frac{(z-1)^{200}}{z^{100}(z-2)^{300}}\, dz$, where the integration path is oriented counterclockwise.

3. Find a one-to-one conformal mapping from the vertical strip $\{ z \in \mathbb{C} : 0 < \operatorname{Re} z < 1 \}$ onto the unit disk $\{ z \in \mathbb{C} : |z| < 1 \}$.

4. Evaluate the improper integral $\displaystyle\int_0^\infty \frac{x \sin x}{1 + x^2}\, dx$.

5. Determine all harmonic functions in the open right half-plane that are constant along each open ray emanating from the origin.

6. Determine the largest possible value of $|f''(0)|$ when $f$ is an analytic function in the disk $D = \{ z \in \mathbb{C} : |z| < 2 \}$ with the property that $\iint_D |f(z)|^2\, dx\, dy \le 3\pi$.

7. Let $f$ be an entire function such that $|f(z^2)| \le e^{|z|}$ for all $z$, and $f(n) = 0$ for each integer $n$. Show that $f$ must be identically equal to zero.

8. (a) Show that there exists an analytic function $f$ in the open right half-plane such that $f(z)^2 + 2f(z) \equiv z^2$.

   (b) Show that your $f$ can be continued analytically to a region containing the set $\{ z \in \mathbb{C} : |z| = 3 \}$.

9. Construct in the open unit disk $\{ z \in \mathbb{C} : |z| < 1 \}$ an analytic function $f$ such that $f(1 - \frac{1}{n^2}) = 0$ for each positive integer $n$, and $f$ has no other zeroes.

10. For *one* of the following, state the theorem and sketch its proof: the big Picard theorem, Mittag-Leffler's theorem, Rouché's theorem.