Complex Analysis Qualifying Examination

May 1996

1. Find a one-to-one conformal mapping that maps the region between the disks $|z| < 1$ and $|z - 1/2| < 1/2$ onto a half-plane.

2. Use residue calculus to evaluate the definite integral

$$\int_0^\infty \frac{\cos x}{(4 + x^2)} dx.$$

3. Let f be an analytic function on the unit disk $|z| < 1$ such that $|f(z)| < 1$, and $f(0) = 1/2$. Find an optimal upper bound on $|f'(0)|$. Give an example to show that your bound is the best possible.

4. Find a maximal region G in the complex plane and two distinct continuous functions f and g defined on G such that $f(z)^2 = g(z)^2 = 1 - z^2$ for all z in G. Are f and g analytic?

5. Suppose f is a nowhere vanishing analytic function in the annulus $1 < |z| < 2$, and for each radius r between 1 and 2, the argument of $f(z)$ is constant on the circle $|z| = r$. Show that f is a constant function.

6. Let u be a continuous function on the closed disk $\bar{B}(a; R)$ that is harmonic in the open disk $B(a; R)$.
   (a) State the Poisson integral representation for u.
   (b) State and prove Harnack's inequality.

7. Find an entire function with double zeros at $0, 1, 2, 3, \cdots$.

8. State and prove the Schwarz Reflection Principle.

9. Prove that the equation $e^z = 1 - z$ has infinitely many solutions.

10. State the Riemann mapping theorem, the Runge approximation theorem, and the argument principle. Sketch the proof of *one* of these three theorems.