Complex Analysis Qualifying Examination

May 1998

1. Let $f(z)$ be doubly periodic with periods 2 and $2i$. Show that if f is entire then it is constant.

2. Show that $z^7 - 5z^3 + 12 = 0$ has all its roots in the annulus $1 < |z| < 2$.

3. For the function $f(z) := \dfrac{1}{z^2 - 3z + 2}$, find the Laurent expansion valid in $1 < |z| < 2$.

4. Find: $\displaystyle\int_0^\infty \frac{\ln(x)}{(1+x)^2} dx$.

5. Find a function u harmonic on the open region between the circles $|z-1| = 1$ and $|z-2| = 2$, continuous on the closure of this region, except at $z = 0$, and satisfying $u = 1$ on the inner circle and $u = 2$ on the outer circle. (Hint: $x + 1$ is harmonic on the strip, $0 < \Re(z) < 1$.)

6. Let $g(z) = e^{iz^2} - e^{2iz}$. Find the genus of g, the order of g, the rank of g. State Hadamard's Factorization Theorem and use it to determine the infinite product for g.

7. Let $D := \{z \in \mathbf{C} : |z| < 1\}$ and $\mathcal{F}$ be the set of all f analytic on D for which $\int_D |f(x+iy)|^2 dxdy \leq 1$. Show that $\mathcal{F}$ is a normal family.

8. Let $D := \{z \in \mathbf{C} : |z| < 1\}$ and let $f : D \to D$ be analytic. If f is one-to-one and onto (bijective), then one can find $a, c \in \mathbf{C}$, with $|c| = 1$ and $|a| < 1$, such that
$$f(z) = c\frac{z-a}{1-\bar{a}z}.$$

9. State and prove a modified version of the Schwarz Reflection Principle in which real line segments are replaced by arcs of circles.

10. State the following theorems and sketch the proof for *one* of them: Monodromy Theorem, Runge's Theorem, Riemann Mapping Theorem, Open Mapping Theorem.