Complex Analysis Qualifying Examination

May 1999

Notation D denotes the open unit disk $\{ z \in \mathbb{C} : |z| < 1 \}$, and $\overline{D}$ denotes the closed unit disk $\{ z \in \mathbb{C} : |z| \leq 1 \}$.

1. Prove that if p is a holomorphic polynomial of degree n, and if $|p(z)| \leq 1$ when $z \in D$, then $|p(z)| \leq |z|^n$ when $z \notin D$.

2. If $0 < a < \dfrac{1}{2}$, how many values of z in D satisfy the equation $\dfrac{z}{a} = 2^z$?

3. If u is a positive harmonic function in D, and $u(0) = 1$, how big and how small can $u(\frac{1}{2})$ be? (A complete solution should provide both a proof of bounds, and examples showing that the bounds are optimal.)

4. Prove that there exists a holomorphic function f defined on the region $\mathbb{C} \setminus \overline{D}$ such that $(f(z))^2 + z^2 = 1$ for all z in the region.

5. Prove that if f is a holomorphic function on an open neighborhood of $\overline{D}$, and if $|f'(z)| < |f(z)|$ when $|z| = 1$, then $f(0) \neq 0$.

6. Prove that if $a \notin \overline{D}$, then $\inf\limits_{c \in \mathbb{C}} \left(\sup\limits_{z \in D} \left| \dfrac{z - c}{z - a} \right| \right) = \dfrac{1}{|a|}$.

7. Use contour integration to prove that $\displaystyle\int_0^\infty \frac{(\log x)^2}{1 + x^2}\, dx = \frac{\pi^3}{8}$.

8. In the following invalid calculation, x denotes a *real* variable. Write an explanation, suitable for a senior undergraduate, of what is wrong.

$$\begin{aligned} (2i \sin(\pi x))^2 &= \left(e^{\pi i x} - e^{-\pi i x}\right)^2 = e^{2\pi i x} - 2 + e^{-2\pi i x} \\ &= \left(e^{2\pi i}\right)^x - 2 + \left(e^{-2\pi i}\right)^x = 1^x - 2 + 1^x \\ &= 1 - 2 + 1 = 0. \end{aligned}$$

9. Prove that every *even* entire function of order 1 has infinitely many zeroes.

10. For *one* of the following, state the theorem and sketch its proof: the monodromy theorem, Jensen's formula, the Riemann mapping theorem.