Differential geometry qualifying exam

January 12, 2007

All answers must be justified for credit (even if it is true/false, multiple choice, etc...)

1. Let $M^2 \subset \mathbb{E}^3$ be a surface, let (x, y) be local coordinates on M centered at a point $p \in M$. Which of the following could be first fundamental forms for M in a neighborhood of p?

   a. $I = dx \circ dx + dy \circ dy$

   b. $I = e^{x+y} dx \circ dx + dy \circ dy$

   c. $I = dx \circ dx - dy \circ dy$

   d. $I = x^2 dx \circ dx + y^2 dy \circ dy$

2. Determine the curvature and torsion functions of the curve $c(t) = (t, 5 + t, t^3)$.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^3$ be given by $f(x^1, \ldots, x^n) = (x^1 + x^2 + \cdots + x^n, x^1x^2 - x^3x^4, x^n)$.

   a. Calculate the rank of f at the point $(1, \ldots, 1)$.

   b. Calculate the rank of f at the point $(1, 0, \ldots, 0)$.

   c. Is $f^{-1}(n, 0, 1)$ a smooth submanifold of $\mathbb{R}^n$?

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x^1, x^2, x^3) = (x^1 + x^2 + x^3, x^1x^2 - x^3, (x^3)^2) = (y^1, y^2, y^3)$.

   a. Calculate $f^*(dy^1)$,

   b. Calculate $f^*(y^2 dy^3)$,

   c. Calculate $f^*(dy^1 \wedge dy^2 \wedge dy^3)$.

   d. At which points $(x^1, x^2, x^3) \in \mathbb{R}^3$ is f not an immersion?

   e. Explain the relationship between parts c. and d. and give a generalization to maps $f : \mathbb{R}^n \to \mathbb{R}^n$.

5. Let V be a vector space. Define V^* (the dual space of V), $V \otimes V$, S^2V and $\Lambda^2 V$. Show that there is a canonical decomposition $V \otimes V = S^2V \oplus \Lambda^2 V$.

6. Consider the surface $M \subset \mathbb{R}^3$ that is the cone with vertex $(1,0,0)$ over a circle of radius R in the (y,z)-plane centered at $(0,0,2)$. (Remove the vertex so it becomes a smooth manifold.)

   a. Give a parametrization of M.

   b. Calculate the Gauss and mean curvature functions for M.

   c. Determine the image of the Gauss map of M in the two-sphere S^2.

7. Let $M^2 \subset \mathbb{R}^3$ be a surface such that for every $x \in M$ there exists a neighborhood $U \subset M$ containing x, such that U is locally isometric to the plane. What are the possible dimensions of the image of the Gauss map of M?