Differential Geometry Qualifying Examination

May 2002

Directions: Work as many problems or parts of problems as you can in the time allotted. Start each numbered problem on a new sheet of paper and write on only one side of each sheet.

Notation: $\mathbb{R}^n$ denotes n-dimensional Euclidean space. M and N are smooth (C^∞) manifolds of dimension m and n, respectively. $C_M^\infty(U)$ denotes the $\mathbb{R}$-algebra of smooth (C^∞) real-valued functions $f\colon\ U \to \mathbb{R}$ on an open set $U \subset M$. $X_M(U)$ denotes the $C_M^\infty(U)$-module of smooth vector fields on U, and $\Omega_M^1(U)$ denotes the $C_M^\infty(U)$-module of smooth one-forms on U.

1. Derive the Frenet formulas for a regular parametrized curve $\alpha\colon\ I \to \mathbb{R}^3$. (You may assume α is parametrized by arc-length.) Explain and justify the steps in your derivation.

2. Consider the surface S in $\mathbb{R}^3$ given parametrically by

$$X(u,v) = ((2+\cos u)\cos v, (2+\cos u)\sin v, -\sin u)$$

for $-\pi < u < 0, 0 < v < \pi$. We orient this surface using the "upward" normal.

   a) Determine the first and second fundamental forms i.e. $E(u,v)$, $F(u,v)$, $G(u,v)$, $e(u,v)$, $f(u,v)$ and $g(u,v)$ in this local coordinate system.
   b) Express the Gaussian curvature K as a function of u, v.
   c) Find the principal curvatures and principal directions at $(0,2,1) \in S$ where $u = \pi/2$, $v = \pi/2$.
   d) Determine the Christoffel symbols Γ_{ij}^k as functions of u and v.
   e) Is the curve $(u(t), v(t)) = \left(-\frac{\pi}{2}, t\right)$, $0 < t < \pi$ a geodesic? (Compute the appropriate covariant derivative.)

3. a) Define what is meant by an affine connection ∇ on a differential manifold M.
   b) If (M,g) is a Riemannian manifold, what properties does the Levi–Civita (or Riemannian) connection $\nabla_{(M,g)}$ satisfy that uniquely characterizes it among all connections?
   c) If $x_1, \ldots, x_n$ are local coordinates on a Riemannian manifold with its Riemannian connection and $\gamma(t) = (x_1(t), x_2(t) \ldots x_n(t))$ is a curve on M write the conditions that $\gamma(t)$ be a geodesic. If $\Gamma_{ij}^k = 0$ for all i, j, and k then which curves in the coordinate patch are geodesics (prove it!)?

4. a) Define what is meant by a partition of unity subordinate to a covering $\{U_\alpha\}$ of M.
   b) Prove that every differentiable manifold M has a Riemannian metric.
   c) Assume M is orientable of dimension n define the volume form ω on M (write it in local coordinates) and show it is a globally well-defined n-form on M.

5. Let M be a Riemannian manifold with Levi-Civita connection ∇. Recall that the curvature tensor R of M can be viewed as assigning to every pair of vector fields $X, Y \in \mathcal{X}(M)$ a mapping $R(X,Y)\colon\ \mathcal{X}(M) \to \mathcal{X}(M)$ where $\mathcal{X}(M)$ is the $C_M^\infty(M)$-module of vector fields on M. We have $R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$.

a) Show that this is $C^\infty(M)$-linear in X, Y and Z.

b) If we define $R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \sum_\ell R^\ell_{ijk}\frac{\partial}{\partial x_\ell}$, then express R^ℓ_{ijk} as functions of $x_1, \ldots, x_n$ in terms of the Christoffel symbols.

6. Let f be a C^∞-function on M, $f\colon\ M \to \mathbb{R}$ and let X, Y be two vector fields on M.

a) Define what we mean by a tangent vector at $p \in M$ and what we mean by a smooth vector field on M.

b) Define Xf and show the resulting function is independent of the choice of coordinates.

c) Define $[X, Y]$ and show it really is a vector field using your definition in a).