Differential Geometry qualifying examination

Directions: Work as many problems or parts of problems that you can in the time alloted. Start each numbered problem on a new sheet of paper and write on only one side of each sheet.

I. Let $c : \mathbb{R} \to \mathbb{E}^3$ be a smooth curve in Euclidean 3-space. Derive expressions in terms of c and its derivatives for the speed, curvature and torsion of c. Characterize the curves with torsion identically zero.

II. Let $c : \mathbb{R} \to \mathbb{E}^2$ be a smooth curve in the Euclidean plane. Consider $\mathbb{E}^2$ as a plane in $\mathbb{E}^3$ and let S denote the surface consisting of the points lying on lines perpendicular to $\mathbb{E}^2$ and containing a point of $c(\mathbb{R})$. Determine the Gauss curvature, mean curvature, and principal curvature functions on S.

III. Let M^n be a smooth n-dimensional manifold and let $\alpha^1, \ldots, \alpha^r \in \Omega^1(M)$. State a necessary and sufficient condition that in the neighborhood of a point $x \in M$ there exists an immersed submanifold $i : N \to M$ of dimension $n - r$ such that $i^*(\alpha^j) = 0$ for $j = 1, \ldots, r$. What is special when $r = n - 1$?

IV. Let V be a real vector space of dimension 3 with basis e_1, e_2, e_3 and let e^1, e^2, e^3 denote the dual basis of V^*.

1. Let $GL(V)$ denote the set of all invertible linear maps $V \to V$. Prove that $GL(V)$ is a Lie group and calculate its dimension.
2. Let $q = e^1 \circ e^1 + e^2 \circ e^2 - e^3 \circ e^3$ be a quadratic form on V. Let $G = \{g \in GL(V) \mid q(g.v, g.w) = q(v, w) \ \forall v, w \in V\}$. Prove that G is a Lie group and calculate its dimension.
3. Determine the Lie algebra of G explicitly as a subalgebra of the space of 3×3 matrices with respect to the given basis of V.

V. Let (M^n, g) be a Riemannian manifold of dimension n with coframing $\eta^1, \ldots, \eta^n$ and connection forms η^i_j.

1. Express the Riemann curvature tensor of M in terms of the coframing, connection forms and their derivatives.
2. Express the Ricci curvature tensor of M in terms of the coframing, connection forms and their derivatives.
3. Let $n = 3$. Show that if the Ricci curvature of M is identically zero then M is locally isometric to $\mathbb{E}^3$.

VI. Let M^n be a differentiable manifold and let $\pi : \mathcal{F}_M \to M$ denote the bundle whose fibers $\pi^{-1}(x)$ (for $x \in M$) consist of all possible bases for T_xM. This problem deals with three different definitions of a connection.

1. Show that there exists a tautological $\mathbb{R}^n$-valued one-form ω on $\mathcal{F}_M$ whose entries span the space of semi-basic forms on $\mathcal{F}_M$. Write

$$\omega = \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}$$

2. Let θ be a $\mathfrak{gl}_n$-valued one-form on $\mathcal{F}_M$ such that $d\omega^i = -\theta^i_j \wedge \omega^j + T^i_{jk}\omega^j \wedge \omega^k$ for some functions $T^i_{jk} : \mathcal{F}_M \to \mathbb{R}$. We call θ a *connection form.*

   Define a *connection operator* on M to be a differential operator $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$, $(X, Y) \mapsto \nabla_X Y$, satisfying the following two properties:

$$\nabla_{fX+Y} Z = f(\nabla_X Z) + \nabla_Y Z$$
$$\nabla_X (fY + Z) = f\nabla_X Y + (Xf)Y + \nabla_X Z$$

   for all $X, Y, Z \in \Gamma(TM)$ and $f \in C^\infty(M)$.

   Show that θ determines a connection operator on M.

3. Call a distribution on $\mathcal{F}_M$ transverse to $\ker \pi_*$ and of complementary dimension, a *connection distribution.* Show that a connection distribution also determines a connection operator on M.

4. Given a connection form θ, show that the distribution $\ker \theta$ is a horizontal distribution and that the connection operator it determines is the same as the one determined directly by θ.

5. State what it means for a connection form to be torsion free.

6. State what it means for a connection operator to be torsion free.

7. Show that a connection form is torsion free iff the connection operator it determines is torsion free.