m is Lebesgue measure on $\mathbb{R}$.

1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued continuous functions on a non empty complete metric space X so that for each $x \in X$, $\liminf_{n\to\infty} f_n(x) > 0$. Prove that there exist $\epsilon > 0$ and a positive integer N so that $\{x \in X : f_n(x) \geq \epsilon \text{ for all } n \geq N\}$ has non empty interior.

2. Prove that the linear span of $\{x^{4n}\}_{n=1}^{\infty}$ is norm dense in $L^2([0,1], m)$.

3. Suppose that $f \in L^1([0,1], m)$ and for all rational numbers a, b for which $0 \leq a \leq b \leq 1$, $\int_a^b f(x)\,dx \geq 0$. Prove that $f \geq 0$ m-a.e.

4. a. State the Closed Graph Theorem.

b. Let X be a Banach space and T a linear transformation from X into X such that $(T - 2I)(T - 5I) = 0$ and both $T - 2I$ and $T - 5I$ have closed ranges. Show that T is a bounded linear transformation.

5. Show that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of polynomials so that $f_n(x) > 0$ for all $x \in [0,1]$, $f_n \to 0$ pointwise, and $\int_{[0,1]} f_n(t)\,dt \geq 1$. What is the minimum value $\int_{[0,1]} \sup_n f_n(t)\,dt$ can be for such a sequence $\{f_n\}_{n=1}^{\infty}$? Justify your assertion.

6. Let X be a non empty compact metric space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $C(X)$. Prove that $f_n \to 0$ weakly in $C(X)$ if and only if $\{f_n\}_{n=1}^{\infty}$ is bounded in $C(X)$ and $f_n \to 0$ pointwise on X.

7. Let $2 < p < \infty$ and let X be a closed linear subspace of $L^p([0,1], m)$. Prove that X is closed in $L^2([0,1], m)$ provided that there exists $M < \infty$ so that for each $f \in X$ for which $\|f\|_p = 1$, $\int_0^1 \min\{|f(t)|, M\}^p\,dt > .0001$.

8. For a function $f : [0,1] \to \mathbb{R}$, define

$$\|f\|_L := |f(0)| + \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : 0 \leq x < y \leq 1\right\}$$

a. Show that the set of all functions f on $[0,1]$ for which $\|f\|_L \le 1$ is a compact subset of $(C[0,1], \|\cdot\|_{\text{sup}})$.

b. Determine whether the (σ-compact) set of all functions f on $[0,1]$ for which $\|f\|_L < \infty$ is dense in $(C[0,1], \|\cdot\|_{\text{sup}})$.

9. Let A and B be compact subsets of $\mathbb{R}$ having positive Lebesgue measure. Prove that $A+B$ has nonempty interior by, for example, doing the following.

(i) Let $f = \chi_A * \chi_B$. Then f is a nonnegative continuous function and

$$0 < \int_{\mathbb{R}} f(x)\,dx < \infty.$$

(ii) If x is such that $f(x) > 0$, then $x \in A+B$.

10. Let $1 < p < \infty$, $p^{-1} + q^{-1} = 1$. For $f \in L_q(0,\infty)$, define $(Tf)(0) := 0$ and $(Tf)(x) := x^{-1/p} \int_0^x f(t)\,dt$ for $0 < x < \infty$. Show that the function Tf is defined, bounded, and continuous on $[0,\infty)$.