

Topology Qualifying Examination

January 2005

Start each problem on a new sheet of paper, and use only one side of each sheet. Write your name and page number in the upper right corner of each page.

Notation. \mathbf{R} denotes the real numbers, and \mathbf{R}^n denotes Euclidean n -dimensional space.

Answer all of questions 1–6.

1. Let \mathbf{R}_ℓ denote the real line \mathbf{R} with the lower limit topology. Prove that \mathbf{R}_ℓ is first countable, Lindelöf, and separable, but not second countable.
2. Let X be a Hausdorff topological space.
 - (a) Prove that X is normal if and only if for any sets U open and C closed with $C \subset U$ there is an open set V with $C \subset V \subset \overline{V} \subset U$.
 - (b) State and prove Urysohn's Lemma.
3. Let X be Hausdorff and paracompact and let $U \subset X \times [0, \infty)$ be an open set that contains $X \times \{0\}$. Show that there is a continuous map $f : X \rightarrow (0, \infty)$ such that $(x, y) \in U$ for all $y \leq f(x)$.
4. Prove that \mathbf{S}^n is not a retract of the disk \mathbf{D}^{n+1} . Use this fact to prove that \mathbf{S}^n is not contractible.
5. Let M be a compact m -manifold and let $\phi : M \rightarrow \mathbf{R}^n$ be smooth.
 - (a) Define *critical value* and *critical point* for ϕ . State Sard's Theorem.
 - (b) Assume $n > 2m + 1$ and ϕ is an embedding. Using ϕ , prove that there exists a smooth embedding $\psi : M \rightarrow \mathbf{R}^{n-1}$. (You may not quote the Whitney Embedding Theorem to prove this result, which is actually part of the proof of the Whitney theorem.)
6. Let X be a topological space with base point x_0 .
 - (a) State the Seifert-Van Kampen Theorem.
 - (b) Let X be the union of the unit sphere \mathbf{S}^2 centered at the origin $x_0 = (0, 0, 0)$ in \mathbf{R}^3 and the three coordinate axes. Find $\pi_1(X, x_0)$.

Answer two problems of questions 7–9. Clearly label the problems that you wish to have graded.

7. Let X be a compact Hausdorff space. Define *universal* net in X . Prove that every universal net in X converges.
8. Let X be a completely regular Hausdorff space. Let \mathbf{F} be the set of all continuous maps $f : X \rightarrow [0, 1]$ and let $Z = \Pi\{[0, 1] : f \in \mathbf{F}\}$. Define $\Phi : X \rightarrow Z$ by $\Phi(x)(f) = f(x)$. Let $\beta(X)$ denote the closure of $\Phi(X)$ in Z .
 - (a) Prove that $\beta(X)$ is compact Hausdorff and $\Phi : X \rightarrow \beta(X)$ is an embedding.
 - (b) Let $f : X \rightarrow [0, 1]$ be a bounded continuous map. Prove that f can be extended uniquely to a continuous map $\beta(X) \rightarrow [0, 1]$.
9. Let G be a (Hausdorff) topological group and let H be a closed normal subgroup. Prove that G/H , with the quotient topology, is a topological group. (Hints. First show that G/H with the topology induced by the canonical map $\pi : G \rightarrow G/H$ is Hausdorff. Next show that π is open and continuous. Finally show that G/H is a topological group.)

Answer one problem of questions 10–11. Clearly label the problem that you wish to have graded.

10. Let X and Y be a topological spaces.
 - (a) Define that X and Y have the *same homotopy type*. A space is *contractible* if it is homotopy equivalent to the one-point space. Prove that X is contractible if and only if the identity map $1_X : X \rightarrow X$ is homotopic to a map $r : X \rightarrow X$ whose image is a single point.
 - (b) Suppose that $Y \subset X$. Define that Y is a *retract* of X . Prove that a retract of a contractible space is contractible.
11. Define a *covering map* from one topological space to another. Show that the projective plane \mathbf{P}^2 has a two-fold cover which is homeomorphic to \mathbf{S}^2 . Prove that the fundamental group of \mathbf{P}^2 is the group \mathbf{Z}_2 of two elements.

Answer one problem of questions 12–13. Clearly label the problem that you wish to have graded.

- 12.** Let X and Y be topological spaces and G is a topological group that acts on X .
- (a) Let $p : X \rightarrow Y$ be a covering map. Define that $D : X \rightarrow X$ is a *deck transformation* of the covering. Explain briefly why the set of deck transformations of p form a group under composition. Define that p is a *regular* covering map.
 - (b) Define that the action of the group G is *properly discontinuous*. Prove that if G acts properly discontinuously on the arcwise connected and locally arcwise connected space X , then $p : X \rightarrow X/G$ is a regular covering map with deck transformation group G .
- 13.** Let M be an n -manifold. Define the *tangent bundle* $T(M)$ of M . Define that M is *parallelizable*. Prove that $\mathbf{S}^n \times \mathbf{R}$ is parallelizable for all n .