TOPOLOGY QUALIFYING EXAM

May 2002

INSTRUCTIONS:

•You must work on all problems below.

•Use a separate sheet of paper for each problem.

Problem 1. Let X be a connected Hausdorff space with at least 2 points. Show that X contains an uncountable number of points. [CORRECTION: X should also be regular.]

Problem 2. Let M and N be submanifolds of $\mathbb{R}^n$.
a) Prove or disprove. If M and N are transverse then $M \cap N$ is a submanifold of $\mathbb{R}^n$.
b) Prove or disprove. If $M \cap N$ is a submanifold of $\mathbb{R}^n$ then M and N are transverse.

Problem 3. Let X and Y be topological spaces.
a) Define the one-point compactification of X.
b) Prove that maps between locally compact Hausdorff spaces are proper if and only if they extend to continuous maps between their one-point compactifications.

Problem 4.
a) Define a completely regular space.
b) State Urysohn's Metrization Theorem.
c) Prove that a closed subspace of a normal space is normal. [Hint: make sure you use the 'closed' hypothesis since this statement is false when closed is removed]

Problem 5. Let M^6 be the 6-manifold $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{S}^2$. Calculate $\pi_1(M^6)$. How many covers does M^6 have? Roughly describe each cover and the subgroup it corresponds with.

Problem 6. Prove that a surjection $f : X \to Y$ is an identification map if and only if (for all functions $g : Y \to Z$ ($g \circ f$ is continuous if and only if g is continuous)). [The parentheses should clarify the statement]

Problem 7. State and prove the (easy) Whitney embedding theorem for compact smooth n-manifolds.

Problem 8. If $f, g : X \to \mathbb{S}^n$ are two maps such that for all x, $f(x) \neq -g(x)$, show that they are homotopic maps. Describe the homotopy you use in words.

Problem 9. Let $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and consider the map $f : \mathbb{S}^1 \to \mathbb{S}^1$ where $f(z) = z^2$. Show that the mapping cylinder M_f is homeomorphic to a Möbius strip.

Problem 10. Let D be the disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$, let S be the circle $\{z \in \mathbb{C} \mid |z| = 1\}$, and let X_n be the space obtained by attaching ∂D to S using the map $z \mapsto z^n$. In other words, the boundary of the disk is wrapped n times around S.
a) Use the Seifert-van Kampen theorem to calculate $\pi_1(X_n)$.
b) Calculate $H_1(X_n)$.
c) Describe in detail the universal cover of X_{18}.