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**Topology Qualifying Exam**  
(May 2003)

**Instructions:** START EACH PROBLEM ON A SEPARATE SHEET OF PAPER AND WRITE ON ONLY ONE SIDE OF THE PAPER. NUMBER EACH PROBLEM ACCORDING TO THE NUMBERING IN THE EXAM.

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**Answer three problems out of the following four problems.**

**Problem 1.** Prove that a compact, Hausdorff space is metrizable if and only if it is second countable. State the main theorems used in your proof.

**Problem 2.** Let  $f : X \rightarrow Y$  be a quotient map, with  $Y$  connected. Show that if  $f^{-1}(\{y\})$  is connected for all  $y \in Y$ , then  $X$  is connected.

**Problem 3.** Prove the following version of Tychonoff's theorem: "A countable product of finite sets (each with the discrete topology) is compact."

**Problem 4.**      **a:** Prove that a paracompact space is normal.

**b:** Prove or give a counterexample: A subspace of a normal space is normal.

**c:** Prove that a locally compact, Hausdorff space is completely regular.

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**Answer three problems out of the following four problems.**

**Problem 5.** Define a smooth manifold.

**a:** Using your definition, prove that a smooth manifold  $X$  is metrizable.

**b:** Explain why  $X$  is paracompact.

**c:** Describe the construction of smooth partitions of unity for a smooth manifold.

**Problem 6.**      **a:** Define a regular value for a smooth map  $f : M \rightarrow N$  between smooth manifolds  $M$  and  $N$  and state Sard's theorem.

**b:** Let  $M$  be a smooth  $m$ -dimensional connected manifold whose fundamental group is finite of odd order. Show that if  $m < n$  then any smooth map  $f : M \rightarrow \mathbb{R}P^n$  is null-homotopic (homotopic to a constant map).

**Problem 7.** Let  $U_n$  denote the set of  $n \times n$  matrices  $A$  with complex entries satisfying  $A \cdot \overline{A}^T = I$ , where  $\overline{A}^T$  denotes the transpose of the conjugate matrix (i.e. the matrix whose entries are the complex conjugate of the entries of  $A$ ). Show that  $U_3$  is a compact smooth manifold and compute its dimension.

**Problem 8.** Let  $S$  be the subspace of  $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$  defined as

$$S := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \mathbf{x} \cdot \mathbf{y} = 0, \|\mathbf{x}\| = 1 \text{ and } \|\mathbf{y}\| = 1\},$$

where  $\mathbf{x} \cdot \mathbf{y}$  denotes the inner (dot) product of  $\mathbf{x}$  and  $\mathbf{y}$ , and  $\|\mathbf{v}\|$  denotes the Euclidean norm of the vector  $\mathbf{v} \in \mathbb{R}^{n+1}$ .

- a:** Show that  $S$  is a compact smooth submanifold of  $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$  and determine its dimension.
- b:** Given  $(\mathbf{x}_0, \mathbf{y}_0) \in S$ , describe the tangent space  $T_{(\mathbf{x}_0, \mathbf{y}_0)}S$  as a linear subspace of  $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ .

**Problem 9.** Let  $X, Y$  and  $Z$  be smooth manifolds with base-points  $x_0, y_0$  and  $z_0$ , respectively.

- a:** Describe the fundamental groups of  $X \vee Y$  and  $X \vee Y \vee Z$  in terms of the fundamental groups of  $X, Y$  and  $Z$ .
- b:** Using the previous item, describe the fundamental group of  $S^1 \vee \mathbb{R}\mathbb{P}^2$ .
- c:** Describe the universal covering space  $\tilde{X}$  of  $X = S^1 \vee \mathbb{R}\mathbb{P}^2$  and the action of group of deck transformations on  $\tilde{X}$ .

**Problem 10.** Let  $f : X \rightarrow Y$  be a covering map, and consider  $x_1, x_2 \in f^{-1}(\{y_0\})$ .

- a:** Show that there is a deck transformation  $\phi : X \rightarrow X$  sending  $x_1$  to  $x_2$  if and only if  $f_{\#}(\pi_1(X, x_1)) = f_{\#}(\pi_1(X, x_2))$ .
- b:** Define a regular covering map and show that a lifting as described in the previous item always exists for a regular covering.

**Problem 11.** Compute the fundamental group of the open subset  $X$  of  $\mathbb{R}^3$  obtained by removing the three coordinate axis.

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