Results on Tight Sampling Sets in Polynomial Spaces

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July 20, 2004

Abstract

Sampling theory is the study of the reconstruction of functions from the set of values they take on at certain points in their domains, and is studied most generally on Hilbert function spaces. The points at which the functions are sampled are known as sets of sampling In this paper, the concept of a tight sampling set (TSS) is introduced, and theorems and conjectures concerning the existence of TSSs in polynomial spaces over intervals in $\mathbb{R}$ are given. It is shown that if a TSS of k elements exists for one such interval, then k element TSSs can be found for any other interval. A characterization of TSSs is polynomial spaces of degree N is given in the form of $2N + 1$ equations, and is used to explore the geometry of TSSs in $P_1[0, 1]$ in considerable detail. Other topics include a preliminary characterization of TSSs in $P_N([0, 1] \times [0, 1])$ along with an example of a TSS in that space, the nonexistence of evenly spaced sampling points in $P_N[a, b]$, and the symmetry of TSSs and the associated frame polynomials.

Sampling theory

The concepts of *sets of sampling* and *frames* are central to the theory of sampling. A set of sampling is a collection (finite or infinite) of points in the domain of a function space such that a function is recoverable from its values at those points.

Definition. *Let* F *be a Hilbert space of functions defined on a domain* D*. Let* $T = \{t_i\}_{i \in I}$ *be a finite or infinite sequence of points in* D*.* T *is said to be a* set of sampling *for* F *if the sampling operator* $S : F \to l^2_{|T|}$ *defined by*

$$S : f \mapsto \begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \end{pmatrix}$$

is bounded (i.e. continuous) and bounded below; i.e. if

$$\exists A, B > 0 \text{ such that } \forall f \in F, A\|f\|^2 \leqslant \sum_i |f(t_i)|^2 \leqslant B\|f\|^2.$$

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[*] In collaboration with Benjamin Aurispa (Texas A&M University) and Sayaka Olsen (University of Nevada, Reno)

Frames are, in a certain sense, to general Hilbert spaces what sampling sets are to function spaces. Specifically, a frame is a collection of vectors such that an arbitrary vector is recoverable from its inner product with the elements of the frame.

Definition. *A frame in a Hilbert space* H *is a collection* $\{x_i\}_{i=1}^{k}$ *of vectors in* H *for which there exist constants* $A, B > 0$ *such that for all* $x \in H$*,*

$$A\|x\|^2 \leqslant \sum_i |\langle x, x_i \rangle|^2 \leqslant B\|x\|^2$$

The constants A, B *are called the lower and upper frame bounds respectively, or the frame constants. When* $A = B$*, the frame is a tight frame, or an* A*-tight frame.*

A definite connection between sets of sampling and frames is given by the Riesz Representation Theorem.

Theorem (Riesz Representation Theorem). *Let* $\phi(x)$ *be a linear functional (i.e.* $\phi : H \to \mathbb{R}$*) on a Hilbert space* H*. There is a unique vector* v_ϕ *such that* $\forall x \in H: \ \phi(x) = \langle x, v_\phi \rangle$*.*

Proof Let $x \in H$ and ϕ be a linear functional on H, and let $\{e_i\}$ be an orthonormal basis of H; define $a_i = \langle x, e_i \rangle$.

$$\phi(x) = \phi(\sum_i a_i e_i) = \sum_i a_i \phi(e_i).$$

Let $v_\phi = \sum_i \overline{\phi(e_i)} e_i$, then $\phi(x) = \langle x, v_\phi$.

$\square$

The Riesz Representation Theorem guarantees that the linear functional of point evaluation at a point t in the domain D, $(p : x \mapsto x(t))$, can be represented by a unique vector v_t in the manner $p(x) = x(t) = \langle x, v_t \rangle$. Consequently, for each set of sampling $\{t_i\}$ there is a corresponding set of vectors $\{v_i\}$ which satisfy the equation

$$\forall f \in F : A\|f\|^2 \leqslant \sum_i |\langle f, v_i \rangle|^2 \leqslant B\|f\|^2;$$

or, each sampling set determines a unique frame (but not vice versa).

Point evaluation vectors in $P_N[0, 1]$

The Riesz Representation Theorem says that vectors which determine to point evaluation functionals exist, but it useful to have a closed-form expression for them in $P_N[0, 1]$. Such an expression has been derived; let $\phi : P_N[0, 1] \to \mathbb{R}$, where $\phi : f \to f(t_0)$, then $\phi(f) = \langle f, g_{t_0} \rangle$, where

$$\begin{aligned} g_{t_0}(t) &= \sum_{i=1}^{N+1}\left[\sum_{j=1}^{N+1}(-1)^{i+j}(i+j-1)\binom{N+1}{N-j+1}\right. \\ &\qquad \left.\binom{N+j}{N-i+1}\binom{i+j-2}{i-1}^2 t_0^{j-1}\right]t^{i-1} \end{aligned}$$

Reconstruction with TSSs

A k-element collection of vectors $\{v_i\}$ is an A-tight frame iff it satisfies the reconstruction formula:

$$\forall f \in F : f = \frac{1}{A}\sum_{i=1}^{k}\langle f, v_i\rangle v_i.$$

If a k-element sampling set $\{t_i\}$ corresponds to an A-tight frame $\{v_i\}$, every function f can be reconstructed from its samples:

$$f = \frac{1}{A}\sum_{i=1}^{k}\langle f, v_i\rangle v_i = \frac{1}{A}\sum_{i=1}^{k} f(t_i)v_i.$$

such a set of sampling is called a *tight set of sampling (TSS)*, and is especially interesting because of the simple reconstruction formula.

Translation, dilation of TSSs

A series of theorems by Benjamin Aurispa establishes that the problem of finding k element TSSs on any polynomial space over an interval is equivalent to finding a k element TSS on $P_N[0,1]$. Proofs are given for the theorems proven this summer.

Theorem. *Fix* $d \in \mathbb{R}$*. Suppose* $\{t_i\}_{i=1}^k$ *is a TSS for* $P_N[a,b]$*. If* $r_i = t_i + d$*, then* $\{r_i\}_{i=1}^k$ *is a TSS for* $P_N[a+d, b+d]$.

Theorem. *Fix* $c \in \mathbb{R},\ c > 0$*. Suppose* $\{t_i\}_{i=1}^k$ *is a TSS for* $P_N[a,b]$*. If* $r_i = ct_i$*, then* $\{r_i\}$ *is a TSS of* $P_N[ca, cb]$.

Corollary. *A TSS of* k*-elements for* $P_N[a,b]$ *where* $a, b \in \mathbb{R}$ *exists iff a* k*-element TSS exists for* $P_N[0,1]$.

Theorem. *Let* $c \in \mathbb{R},\ c > 0$ *and suppose that* $\{g_i\}_{i=1}^k$ *is a tight frame for* $P_N[a,b]$ *corresponding to a tight sampling set* $\{t_i\}_{i=1}^k$*. Set* $h_i(x) = \frac{1}{c}g_i\left(\frac{1}{c}x\right)$*, then* $\{h_i\}_{i=1}^k$ *is a tight frame for* $P_N[ca, cb]$ *with corresponding TSS* $\{ct_i\}_{i=1}^k$.

Proof Let $p \in P_N[ca, cb]$, then

$$\langle p, h_i\rangle = \int_{ca}^{cb} p(x)\frac{1}{c}g_i\left(\frac{1}{c}x\right)dx = \int_a^b p(cu)g_i(u)du = p(ct_i).$$

By the previous theorems,

$$\forall p \in P_N[ca, cb] : \sum_{i=1}^{k} |\langle p, h_i\rangle|^2 = A\|p\|^2$$

□

Theorem. *Let* $d \in \mathbb{R}$ *and suppose that* $\{g_i\}_{i=1}^k$ *is a tight frame in* $P_[a, b]$ *corresponding to a tight sampling set* $\{t_i\}_{i=1}^k$. *Set* $h_i(x) = g_i(x - d)$. *Then* $\{h_i\}_{i=1}^k$ *is a tight frame for* $P_N[a + d, b + d]$ *corresponding to the TSS* $\{t_i + d\}_{i=1}^k$.

Proof Let $p \in P_N[a + d, b + d]$, then

$$\langle p, h_i\rangle = \int_{a+d}^{b+d} p(x)g_i(x - d)dx = \int_a^b p(u + d)g_i(u)du = p(t_i + d)$$

Then, by the previous theorems,

$$\forall p \in P_N[a + d, b + d] : \sum_{i=1}^{k} |\langle p, h_i\rangle|^2 = \sum_{i=1}^{k} |p(t_i + d)|^2 = A\|p\|^2.$$

□

Characteristic equations for k-element TSSs in $P_N[0, 1]$

Let $\{t_i\}$ be a TSS and let $f = \sum_{j=0}^N a_j t^j$ be an arbitrary element of $P_N[0, 1]$.

$$\begin{aligned}
\|f\|^2 &= \int_0^1 \left(\sum_{j=0}^{N} a_j t^j\right)\left(\sum_{j'=0}^{N} a_{j'} t^{j'}\right) dt \\
&= \sum_{j=0}^{N}\sum_{j'=0}^{N} a_{j'} a_j \int_0^1 t^{j+j'} dt \\
&= \sum_{j=0}^{N}\sum_{j'=0}^{N} \frac{a_j a_{j'}}{j + j' + 1}
\end{aligned}$$

$$\sum_{i=1}^{k} |f(t_i)|^2 = \sum_{i=1}^{k}\sum_{j=0}^{N}\sum_{j'=0}^{N} a_j a_{j'} t_i^{j+j'}$$

Since $\{t_i\}$ is a TSS, $\|f\|^2 = A\sum_{i=1}^k |f(t_i)|^2$:

$$\sum_{i=1}^{k}\sum_{j=0}^{N}\sum_{j'=0}^{N} a_j a_{j'} t_i^{j+j'} = A \sum_{j=0}\sum_{j'=0} \frac{a_{j'} a_j}{j + j' + 1}.$$

This is true for all f, so $\forall s \in \{0, \ldots, 2N\}:$

$$\sum_{i=0}^{k} t_i^s = \frac{A}{s+1}.$$

Therefore only if a k-element set of points $\{t_i\}$ satisfies the above $2N+1$ equations and all $t_i \in [0,1]$, is it a TSS.

Example: equations for $P_2[0,1]$, $k = 4$

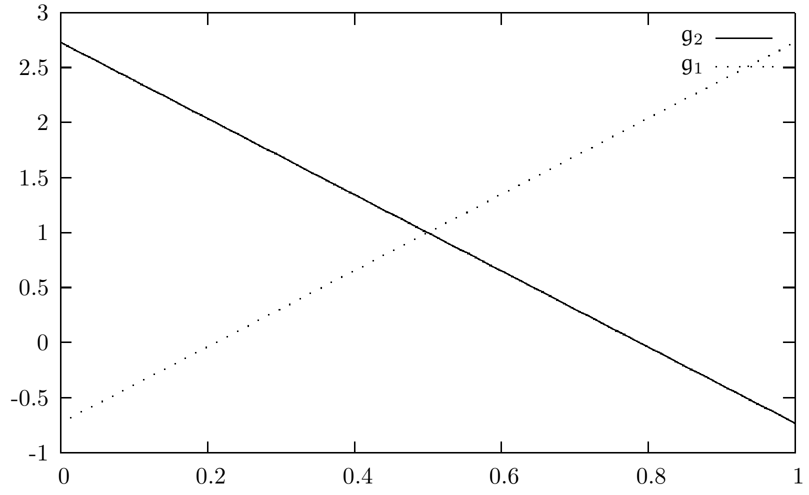
$$\begin{aligned} 1+1+1+1 &= A \\ t_1+t_2+t_3+t_4 &= \frac{A}{2} \\ t_1^2+t_2^2+t_3^2+t_4^2 &= \frac{A}{3} \\ t_1^3+t_2^3+t_3^3+t_4^3 = \frac{A}{4} t_1^4+t_2^4+t_3^4+t_4^4 &= \frac{A}{5} \end{aligned}$$

Immediately, it is seen that $A = k$; also there are no Parseval sampling sets in $P_N[0,1]$. Also, if a k-element TSS does not exist for $P_{N_0}[0,1]$, then no k-element TSSs exist for $P_M[0,1]$ where $M > N_0$. The contrapositive is also true: if $\{t_i\}$ is a k-element TSS for $P_{N_0}[0,1]$, then it is also a k-element TSS for $P_M[0,1]$, where $M < N_0$.

Examples of TSSs

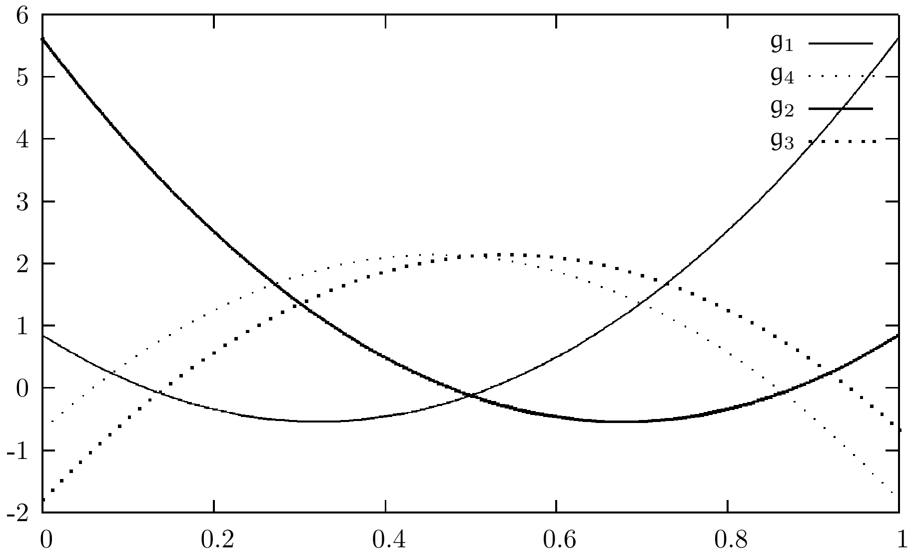
$P_1[0,1]$, $k = 2$

$$\begin{aligned} t_1 &= \tfrac{3+\sqrt{3}}{6} \qquad & g_1 &= 2\sqrt{3}t - \sqrt{3} + 1 \\ t_2 &= \tfrac{3-\sqrt{3}}{6} \qquad & g_2 &= -2\sqrt{3}t + \sqrt{3} + 1 \end{aligned}$$



$P_2[0,1],\ k = 4$

$$
\begin{aligned}
t_1 &= \tfrac{1}{2} + \tfrac{\sqrt{15(2\sqrt{5}+5)}}{30}\\
t_2 &= \tfrac{1}{2} - \tfrac{\sqrt{15(2\sqrt{5}+5)}}{30}\\
t_3 &= \tfrac{1}{2} + \tfrac{\sqrt{-15(2\sqrt{5}-5)}}{30}\\
t_4 &= \tfrac{1}{2} - \tfrac{\sqrt{-15(2\sqrt{5}-5)}}{30}\\
g_1 &= 6\sqrt{5}t^2 + \left(\frac{2\sqrt{15(2\sqrt{5}+5)}}{5} - 6\sqrt{5}\right)t - \frac{\sqrt{15(2\sqrt{5}+5)}}{5} + \sqrt{5} + 1\\
g_2 &= 6\sqrt{5}t^2 + \left(\frac{-2\sqrt{15(2\sqrt{5}+5)}}{5} - 6\sqrt{5}\right)t + \frac{\sqrt{15(2\sqrt{5}+5)}}{5} + \sqrt{5} + 1\\
g_3 &= -6\sqrt{5}t^2 + \left(\frac{2\sqrt{-15(2\sqrt{5}-5)}}{5} + 6\sqrt{5}\right)t - \frac{\sqrt{-15(2\sqrt{5}-5)}}{5} - \sqrt{5} + 1\\
g_4 &= -6\sqrt{5}t^2 + \left(\frac{-2\sqrt{-15(2\sqrt{5}-5)}}{5} + 6\sqrt{5}\right)t + \frac{\sqrt{-15(2\sqrt{5}-5)}}{5} - \sqrt{5} + 1
\end{aligned}
$$



Solution sets for $k \geqslant 2$ in $P_1[0,1]$ always exist

The equations which must be satisfied in this case are:

$$\sum_{i=1}^{k} t_i = k/2 \text{ and } \sum_{i=1}^{k} t_i^2 = k/3$$

To solve them, rotate the coordinate system so the normal vector of the plane coincides with a coordinate axis:

$$(1,\ldots,1) \overset{R}{\mapsto} (\sqrt{k}, 0, \ldots, 0)$$

$$\langle (t_1,\ldots,t_k),(1,\ldots,1)\rangle = \frac{k}{2} \overset{R}{\leftrightarrow} \langle (t_1',\ldots,t_k'),(\sqrt{k},0,\ldots,0)\rangle = \frac{k}{2}$$

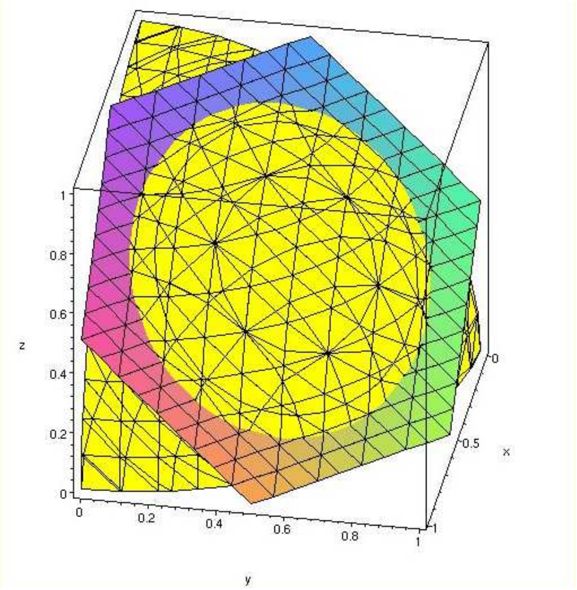
$$\Rightarrow t'_1 = \frac{\sqrt{k}}{2} \text{ and } \sum_{i=2}^{k} {t'_i}^2 = \frac{k}{12}$$

the solution set is a circle with center $(\frac{\sqrt{k}}{2}, 0, \ldots, 0) = \frac{1}{2}(\sqrt{k}, 0, \ldots, 0)$, or in the original coordinate system, $(1/2, 1/2, \ldots, 1/2)$. The radius is $\sqrt{\frac{k}{12}}$. Therefore, solutions always exist.

Finding the 3 element TSSs in $P_1[0, 1]$

These TSSs must satisfy the conditions

$$\sum_{i=1}^{3} t_i = \frac{3}{2} \text{ and } \sum_{i=1}^{3} t_i^2 = 1.$$



To solve them, rotate the coordinate axes so the normal vector of the plane lines up with the t'_1-axis.

$$\underbrace{\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \end{pmatrix}}_{R} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\vec{n}} = \underbrace{\begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix}}_{\vec{n}'}$$

The equation for the plane is now $t'_1 = \frac{\sqrt{3}}{2}$. And the solution set to the equations is a circle centered at $\langle \frac{\sqrt{3}}{2}, 0, 0\rangle$ in the new coordinate system with radius $\frac{1}{2}$. Therefore,

$$\underbrace{\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{6}}{3} \end{pmatrix}}_{R^{-1}} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}\cos\theta \\ \frac{1}{2}\sin\theta \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{2}}{4}\cos\theta - \frac{\sqrt{6}}{12}\sin\theta \\ \frac{1}{2} + \frac{\sqrt{2}}{4}\cos\theta - \frac{\sqrt{6}}{12}\sin\theta \\ \frac{1}{2} + \frac{\sqrt{6}}{6}\sin\theta \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

is a parameterization of the solution set. For all θ, the solution vector lies within the $[0,1]$ cube, so any solution corresponds to a TSS.

Example: $\theta = \frac{\pi}{2}$

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{6}}{12} \\ \frac{1}{2} - \frac{\sqrt{6}}{12} \\ \frac{1}{2} + \frac{\sqrt{6}}{6} \end{pmatrix}$$

The 4 element TSSs in $P_1[0,1]$

Using the same method as for $k = 3$, the parameterization for $k = 4$ is

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{2}}{2\sqrt{3}}\sin\phi\cos\theta - \frac{1}{2\sqrt{3}}\sin\phi\cos\theta \\ \frac{1}{2} + \frac{\sqrt{2}}{2\sqrt{3}}\sin\phi\cos\theta - \frac{1}{2\sqrt{3}}\sin\phi\cos\theta \\ \frac{1}{2} + \frac{1}{2\sqrt{3}}\sin\phi\sin\theta - \frac{\sqrt{2}}{2\sqrt{3}}\cos\phi \\ \frac{1}{2} + \frac{1}{2\sqrt{3}}\sin\phi\sin\theta + \frac{\sqrt{2}}{2\sqrt{3}}\cos\phi \end{pmatrix}$$

Again, for all choices of $\phi \in [0, \pi], \theta$, the solution vector lies within the $[0,1]$ hypercube, so all solutions of the characteristic equations determine a TSS.

However, for $k = 5$ and $k = 6$, some solutions of the characteristic equations fall outside of the $[0,1]$ cube.

Non-existence of evenly spaced TSSs in $P_N[0,1]$

Evenly spaced TSSs do not exist for $P_N[0,1]$.

Proof Assume there exists such a TSS containing k elements; i.e., there is a partition of $[0,1]$ into $k-1$ intervals of length $\frac{1}{k-1}$. Then $t_s = \frac{s-1}{k-1}$, and from the conditions for a TSS:

$$\begin{aligned}
\frac{k}{3} &= \sum_{s=1}^{k} \left(\frac{s-1}{k-1}\right)^2. \\
\frac{k(k-1)^2}{3} &= \sum_{s=1}^{k} s^2 - 2\sum_{s=1}^{k} s + k \\
\frac{k^3-2k^2+k}{3} &= \frac{k(k+1)(2k+1)}{6} - \frac{2k(k+1)}{2} + k \\
2k^3-4k^2+2k &= 2k^3-3k^2+k \\
k^2-k &= 0
\end{aligned}$$

The only solutions are $k=0$ and $k=1$, degenerate cases. □

Characteristic equations for TSSs in $P_N([0,1]\times[0,1])$

For $\{t_i = (x_i, y_i)\}$ to be a k-element TSS, $\forall s \in 0,\ldots,2N \wedge \forall j \in 0,\ldots,..., N \wedge \forall i \in 0,\ldots,N$

$$\begin{array}{l}
\sum_{d=1}^{k} x_d^s y_d^{2N-s} = \frac{A}{(s+1)(2N-s+1)} \\
\sum_{d=1}^{k} x_d^s = \frac{A}{s+1} \\
\sum_{d=1}^{k} y_d^s = \frac{A}{s+1} \\
\sum_{d=1}^{k} x_d^s y_d^{N-j} = \frac{A}{(s+1)(N-j+1)} \\
\sum_{d=1}^{k} x_d^{i+j} y_d^{N-j} = \frac{A}{(i+j+1)(N-j+1)} \\
\sum_{d=1}^{k} x_d^i y_d^j = \frac{A}{(i+1)(j+1)}
\end{array}$$

From these equations, it can be seen that for $\{t_i\}$ to be a TSS in $P_N([0,1]\times[0,1])$, $\{x_i\}$ and $\{y_i\}$ must be TSSs in $P_N[0,1]$. However, the converse may not be true, because of the equations involving mixed terms. Also, in a manner similar to TSSs over $P_N[0,1]$, if a k-element TSS does not exist for $P_{N_0}([0,1]\times[0,1])$, then no k-element TSSs exist for $P_M([0,1]\times[0,1])$ where $M > N_0$. Also, if $\{t_i\}$ is a k-element TSS for $P_{N_0}([0,1]\times[0,1])$, then it is also a k-element TSS for $P_M([0,1]\times[0,1])$, where $M < N_0$.

One special 4-element TSS for $P_1([0,1]\times[0,1])$ is the Cartesian product of the 2-element TSS for $P_1[0,1]$; i.e. let $t_1 = \frac{3+\sqrt{3}}{6}$ and $t_2 = \frac{3-\sqrt{3}}{6}$, then

$$\{(t_1,t_1),\ (t_1,t_2),\ (t_2,t_1),\ (t_2,t_2)\}$$

is a TSS for $P_1([0,1]\times[0,1])$.

Open Questions

- Given k, N what portion of the solution set of k-tuples $(\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_\mathsf{k})$ lies in the cube $\underbrace{[0,1] \times [0,1] \times \cdots \times [0,1]}_{\mathsf{k} \text{ times}}$, thus forming TSSs?
- Conjecture: A TSS of $\mathsf{P}_\mathsf{N}[0,1]$ contains at least $2\mathsf{N}$ elements.
- Is the given example of a 4-element TSS in $\mathsf{P}_2[0,1]$ the only one?
- Conjecture: Given k, N, there is either exactly one k element TSS or uncountably many $\mathsf{P}_\mathsf{N}[0,1]$.
- Examine the distribution of the points in a TSS (for symmetries, etc.)