2007 High School Math Contest
Power Team Exam

1. Two positive numbers, $a < b$, are said to be in the Golden Ratio if the ratio of their sum to the larger equals the ratio of the larger to the smaller. That is,

$$\frac{a+b}{b} = \frac{b}{a}.$$

This ratio is commonly denoted by the Greek letter ϕ.

a. Find the exact value of this ratio.

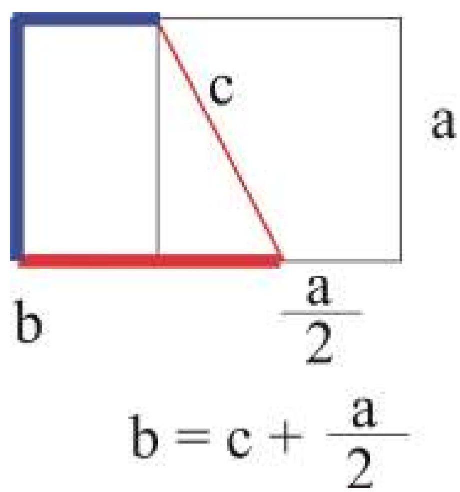
b. All real numbers have what is called a continued fraction representation. In particular, for any positive real number x there is a sequence $a_0, a_1, \cdots, a_n, \cdots$ of positive integers (a_0 may be 0) such that

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

Find a continued fraction representation for the Golden Ratio ϕ.

c. Show that if $0 < a < b$ and they are in the Golden Ratio, then $0 < b - a < a$ and these later two numbers are also in the Golden Ratio.

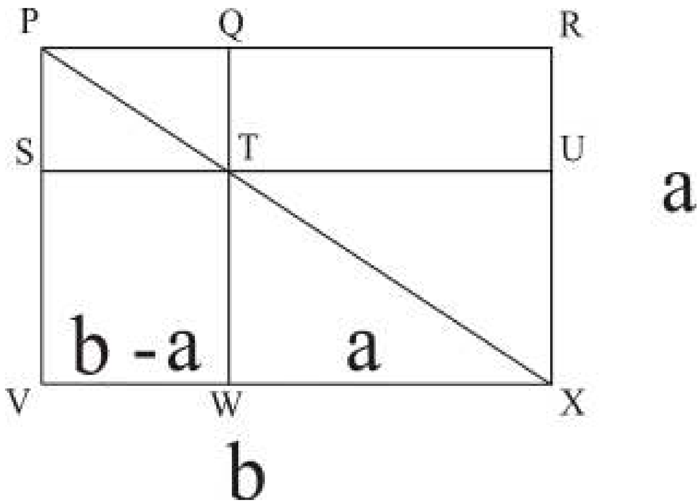
2. A rectangle is said to be a Golden Rectangle if its sides are in the Golden Ratio. Show that the following construction leads to a Golden Rectangle. Take a square of side length a. Draw a straight line from the midpoint of one of the sides to one of the vertices on the opposite side. Denote the length of that line by c. (See the figure below.)



Show that the rectangle constructed above with sides of length a and b is a Golden Rectangle.

In the following a Golden Rectangle is said to be in the horizontal position if its base is the longer side, and in the vertical position if its base is the shorter side. In the figure above, the constructed Golden Rectangle is in the horizontal position.

3. Consider the Golden Rectangle in the horizontal position shown below.



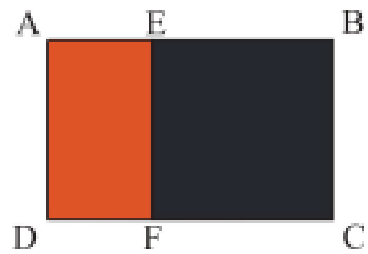
The interior vertical line partitions the original rectangle into a square (QRXW) with sides of length *a*, and a rectangle (PQWV). Once this vertical line is drawn the diagonal line PX is drawn, and then the interior horizontal line STU. These three lines partition the Golden Rectangle into 8 different sub-rectangles.

a. Show STWV is a square.

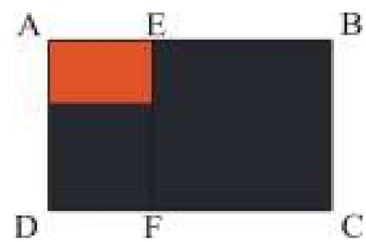
b. Show PQWV is a Golden Rectangle.

c. How many more, if any, of the sub-rectangles besides PQWV are also Golden Rectangles? In your answer use the letters P, Q, ⋯, X to denote the rectangle you are referring to. For example the small rectangle in the upper left corner is rectangle PQTS. To answer this question, construct a table containing for each of the eight rectangles the ratio of the rectangle's longer side to its shorter expressed in terms of *a* and *b* first, and then in terms of the Golden Ratio ϕ.

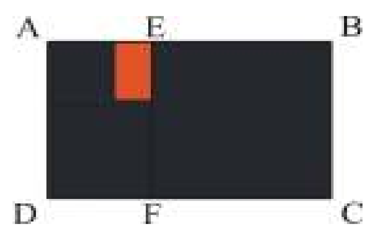
4. ABCD, shown below, is a Golden Rectangle in a horizontal position. AEFD, the orange colored region obtained by removing the square EBCF, is, as we saw previously, a Golden Rectangle in a vertical position.



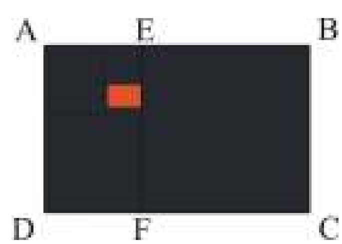
From the Golden Rectangle AEFD remove the square from its lower region. The resulting figure, shown in orange is also a golden Rectangle. (Be sure to explain why this is true.)



From this Golden Rectangle remove the square in its left region, producing the Golden Rectangle shown below in orange.



From this last Golden Rectangle remove a square from its top region, producing once more a Golden Rectangle. See figure below.



Now imagine continuing the process indefinitely. Each of the new Golden Rectangles is contained in the previous Golden Rectangle, the edge lengths of these Golden Rectangles are approaching zero. It can be shown that there is exactly one point contained in all of these golden Rectangles. What are the coordinates of this unique point, and express these coordinates in terms of the Golden Ratio ϕ and the side lengths a and b, where $0 < a < b = \phi a$.