

## Solutions 2007 Power Team Exam

1. Two positive numbers,  $a < b$ , are said to be in the Golden Ratio if the ratio of their sum to the larger equals the ratio of the larger to the smaller. That is,

$$\frac{a+b}{b} = \frac{b}{a}.$$

This ratio is commonly denoted by the Greek letter  $\phi$ .

- a. Find the exact value of this ratio.

From  $\frac{a+b}{b} = \frac{b}{a}$  we have  $a^2 + ab - b^2 = 0$ , which leads to

$$a^2 \left( \left( \frac{b}{a} \right)^2 - \frac{b}{a} - 1 \right) = a^2 (\phi^2 - \phi - 1) = 0.$$

Thus,  $\phi = \frac{1 \pm \sqrt{5}}{2}$ , since  $\phi > 1$ , we have

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

- b. Find a continued fraction representation for the Golden Ratio  $\phi$ .

From  $\phi^2 - \phi - 1 = 0$  we have  $\phi = 1 + \frac{1}{\phi}$ . Thus,

$$\begin{aligned} \phi &= 1 + \frac{1}{\phi} \\ &= 1 + \frac{1}{1 + \frac{1}{\phi}} \\ &= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}} \\ &\quad \dots \\ &= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} \end{aligned}$$

That is, all of the  $a_n$  in the continued fraction expansion of  $\phi$  equal 1.

- c. Show that if  $0 < a < b$  and they are in the Golden Ratio, then  $0 < b - a < a$  and these later two numbers are also in the Golden Ratio.

Since  $\phi = \frac{1 + \sqrt{5}}{2} < 2$  we have

$$\phi - 1 < 1$$

$$a\phi - a < a$$

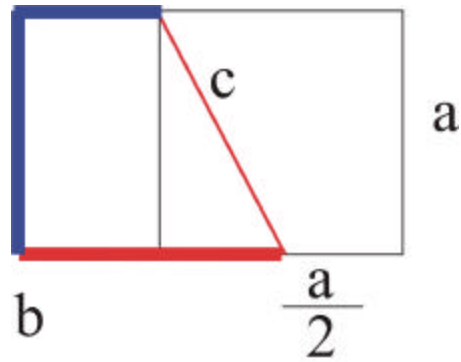
$$b - a < a.$$

Moreover, we know that  $b^2 - ab - a^2 = 0$ , which implies

$$a^2 = b^2 - ab = b(b - a)$$

$$\frac{a}{b-a} = \frac{b}{a} = \phi.$$

2. A rectangle is said to be a Golden Rectangle if its sides are in the Golden Ratio. Show that the following construction leads to a Golden Rectangle. Take a square of side length  $a$ . Draw a straight line from the midpoint of one of the sides to one of the vertices on the opposite side. Denote the length of that line by  $c$ . (See the figure below.)



$$b = c + \frac{a}{2}$$

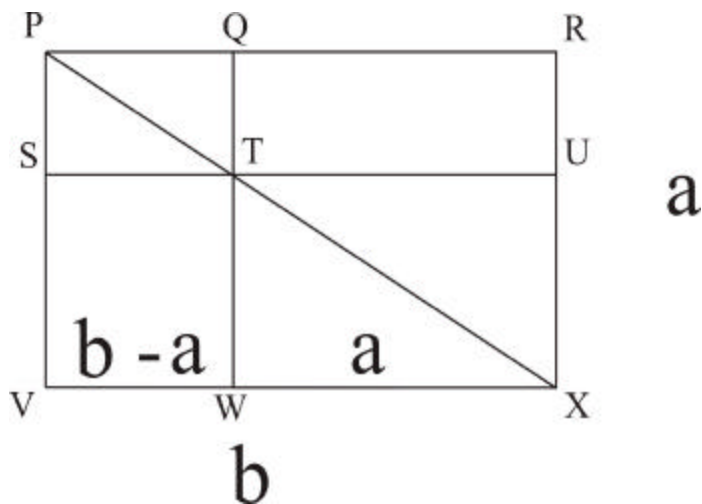
Show that the rectangle constructed above with sides of length  $a$  and  $b$  is a Golden Rectangle.

In the following a Golden Rectangle is said to be in the horizontal position if its base is the longer side, and in the vertical position if its base is the shorter side. In the figure above, the constructed Golden Rectangle is in the horizontal position.

The length  $c$  equals  $\sqrt{(a/2)^2 + a^2} = a \frac{\sqrt{5}}{2}$ . Thus, the ratio of the longer side to the smaller side of this rectangle is

$$\begin{aligned} \frac{b}{a} &= \frac{a/2 + (a/2)\sqrt{5}}{a} \\ &= \frac{1 + \sqrt{5}}{2} = \phi. \end{aligned}$$

3. Consider the Golden Rectangle in the horizontal position shown below.



The interior vertical line partitions the original rectangle into a square (QRXW) with sides of length  $a$ , and a rectangle (PQWV). Once this vertical line is drawn the diagonal line  $PX$  is drawn, and then the interior horizontal line  $STU$ . These three lines slice the Golden Rectangle into 8 different sub-rectangles.

a. Show  $STWV$  is a square.

Since  $STWV$  is a rectangle to see that it is a square it will suffice to show that  $TW$  has length equal to  $b - a$ . Triangles  $PXV$  and  $TXW$  are similar, so we have

$$\frac{TW}{PV} = \frac{WX}{VX} = \frac{a}{b}$$

$$TW = \frac{a}{b}a = \frac{a^2}{\phi a} = \frac{a}{\phi} = b - a.$$

Since,  $VW$  also has length  $b - a$ ,  $STWV$  is a square.

b. Show  $PQWV$  is a Golden Rectangle.

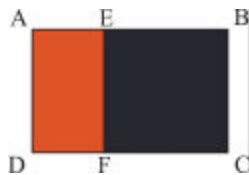
We saw in 1c. that  $b - a$  and  $a$  are in the Golden Ratio. Thus,  $PQWV$  is a golden rectangle.

c. How many more, if any, of the sub-rectangles besides  $PQWV$  are also Golden Rectangles? In your answer use the letters  $P, Q, \dots, X$  to denote the rectangle you are referring to. For example the small rectangle in the upper left corner is rectangle  $PQTS$ . To answer this question, construct a table containing for each of the eight rectangles the ratio of the rectangle's longer side to its shorter expressed in terms of  $a$  and  $b$  first, and then in terms of the Golden Ratio  $\phi$ .

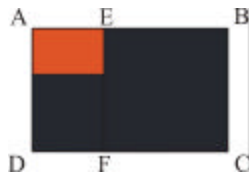
Rectangle	PQWV	QRXW	PRUS	SUXV	PQTS	STWV	QRUT	TUXW
Side Ratios	$\frac{a}{b-a}$	$\frac{a}{a}$	$\frac{b}{2a-b}$	$\frac{b}{b-a}$	$\frac{b-a}{2a-b}$	$\frac{b-a}{b-a}$	$\frac{a}{2a-b}$	$\frac{a}{b-a}$
	$\phi$	$\phi^0$	$\phi^2 / (\phi - 1) = \phi^3$	$\phi^2$	$\phi$	$\phi^0$	$\phi^2$	$\phi$

Thus, there are three Golden Rectangles:  $PQWV$ ,  $PQTS$ , and  $TUXW$ .

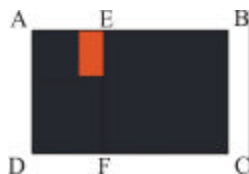
4. ABCD, shown below, is a Golden Rectangle in a horizontal position. AEFD, the orange colored region obtained by removing the square EBCF, is, as we saw previously, a Golden Rectangle in a vertical position.



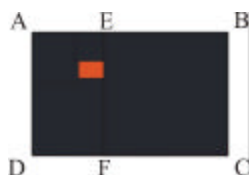
From the Golden Rectangle AEFD remove the square from its lower region. The resulting figure, shown in orange is also a golden Rectangle. (Be sure to explain why this is true.)



From this Golden Rectangle remove the square in its left region, producing the Golden Rectangle shown below in orange.



From this last Golden Rectangle remove a square from its top region, producing once more a Golden Rectangle. See figure below.



Now imagine continuing the process indefinitely. Each of the new Golden Rectangles is contained in the previous Golden Rectangle, the edge lengths of these Golden Rectangles are approaching zero. It can be shown that there is exactly one point contained in all of these golden Rectangles. What are the coordinates of this unique point, and express these coordinates in terms of the Golden Ratio  $\phi$  and the side lengths  $a$  and  $b$ , where  $0 < a < b = \phi a$ .

The answer is easy to compute. The point that lies in each of the Golden Rectangles is the point of intersection of the two lines AC and DE. The proof of this will follow after we compute the coordinates of this point.

Assuming the origin is at the corner D of the golden rectangle, then the line DE, which has slope  $\phi$ , (since DAEF is a Golden Rectangle from problems 2 and 3) has equation  $y = \phi x$ , while the second line AC has slope  $-1/\phi$ , and hence has equation  $y = -x/\phi + a$ . Setting these two equal to each other we have

$$\begin{aligned} \phi x &= -\frac{x}{\phi} + a \\ x &= \frac{a}{\phi + 1/\phi} = \frac{\phi a}{\phi^2 + 1} = \frac{b}{\phi^2 + 1} = \frac{a^2 b}{a^2 + b^2} \\ y &= \phi x = \frac{\phi^2 a}{\phi^2 + 1} = \frac{\phi b}{\phi^2 + 1} = \frac{ab^2}{a^2 + b^2} \end{aligned}$$

Now for the harder part of this problem. The key observation is to note that when we discarded the first square we were left with a Golden Rectangle in the vertical position, and when we discarded the square from it, we used the line DE. Moreover after a total of four squares are discarded we have a Golden Rectangle in the horizontal position with its lower left and lower right corners in exactly the same position with respect to the lines DE and AC as the very first Golden Rectangle, which means the intersection of the two lines lies inside this Golden rectangle. Repeating the process another 4 times leaves us with the same observation. So the point found above must be the unique point that belongs to each of the Golden Rectangles.