

Best Student Exam

TAMU High School Mathematics Contest

November 18, 2000

1. Simplify the expression

$$(x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 1$$

2. Find the minimum value of $\sqrt{x^2 + y^2}$ if $6x - 5y = 4$.
3. Suppose $x, b > 0$ and

$$\log_{b^2} x + \log_{x^2} b = 1$$

Find x .

4. The sum of n terms in an arithmetic progression is 153, and the common difference is 2. If the first term is an integer, and $n > 1$, then what is the number of all possible values for n ?
5. Let f be a function such that $f(3) = 1$ and

$$f(3x) = x + f(3x - 3)$$

for all x . Find $f(300)$.

6. Suppose $\triangle ABC$ is an equilateral triangle and P is a point interior to $\triangle ABC$. If the distance from P to sides AB , BC and AC is 6, 7 and 8 units respectively, what is the area of $\triangle ABC$?
7. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ then the product $A \cdot B$ is defined to be

$$AB = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}$$

Furthermore, $A^2 = A \cdot A$, $A^3 = A \cdot A \cdot A$, etc... If $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ find A^{241} .

8. A circle and a parabola are drawn in the xy -plane. The circle has its center at $(0, 5)$ with a radius of 4, and the parabola has its vertex at $(0, 0)$. If the circle is tangent to the parabola at two points, give the equation of the parabola.

9. The triangle PQR sits in the xy -plane with $P = (0, 0)$, $Q = (3, 12)$ and $R = (6, 0)$. Suppose the x -axis represents the horizontal ground and the triangle is rotated counter clockwise around the origin (note that P will stay fixed) until it reaches a position where it balances perfectly on the vertex P . What is the y -coordinate of the point Q when the triangle is balanced?
10. A circle is placed in the xy -plane and a line L is drawn through the center of the circle. Suppose P is a point interior to the circle which is 6 units from the circle, 6 units from the line L and 10 units from the closest intersection point of the line L with the circle. What is the area of the circle?
11. Five people are asked (individually) to choose a random integer in the interval $[1, 20]$. What is the probability that everyone chooses a different number?
12. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is said to be singular if $ad - bc = 0$. If the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is created at random by choosing integer values a, b, c, d at random from the interval $[-3, 3]$, what is the probability that the matrix will be singular?
13. Consider the table of values shown below

2	a	b	2
c	2	2	d
e	2	2	f
2	g	h	2

- All rows and columns of this table sum to 0. In addition, $a + c = 5$ and $eg = 22$. Find all possible solutions (a, b, c, d, e, f, g, h) .
14. Let $P > 0$ and suppose $\triangle ABC$ is an isosceles right triangle with area P square inches. What is the radius of the circle that passes through the points A , B and C ?
15. How must the numbers a , b and c be related for the following system to have at least one solution?

$$\begin{aligned}x - 2y + z &= a \\2x + y - 2z &= b \\x + 3y - 3z &= c\end{aligned}$$

16. Let x be a real number and create a triangle having vertices $(-2, 1)$, $(1, 3)$ and $(3x, 2x - 3)$. Give a formula for the area of this triangle.

17. The final race in a swimming event involves 8 swimmers. Three of the swimmers are from one country and the other five are from different countries. Each is to be given a lane assignment between 1 and 8 for the race. Aside from the obvious rule that no two swimmers can be assigned to the same lane, there are two other restrictions. The first is that no two swimmers from the same country can be in adjacent lanes. The second is that the two outside lanes cannot be occupied by swimmers from the same country. With these rules, how many different lane assignments are possible for this race?
18. Let $r > 0$. Four circles of radius $2r$ are placed in the xy -plane so that their centers are located at $(-r, -r)$, $(-r, r)$, (r, r) and $(r, -r)$. What is the area of the region of intersection of these circles?