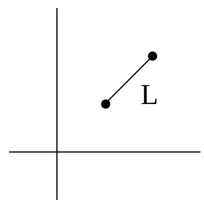
CD EXAM

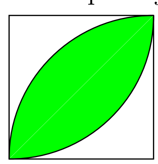
HIGH SCHOOL MATH CONTEST

1. Problems

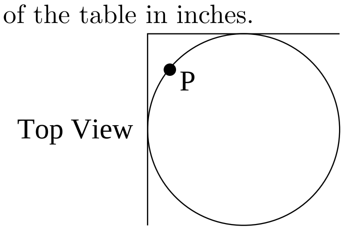
1. Let L be the line segment from $(1, 1)$ to $(2, 2)$. Let S be the set of all points in the xy-plane that are within a distance 1 from the line segment L. What is the area of S?



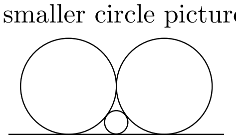
2. In the square shown, each curve is an arc of a circle with radius 4, having centers at the vertices of the square. Find the number of square units of area of the shaded portion of the figure. Express your answer in terms of π.



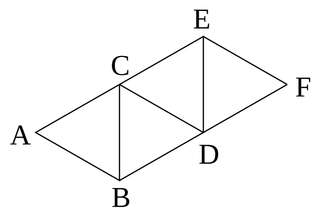
3. A circular table is pushed into the corner of a square room so that the point P on the edge of the table is 8 inches from one wall and 9 inches from the other. Find the radius of the table in inches.



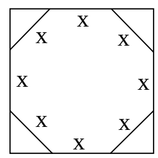
4. Given two circles of radius 1 with a common tangent line and a common point, find the radius of the smaller circle pictured below.



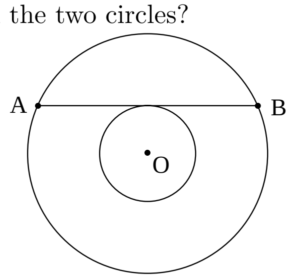
5. Four equilateral triangles with side length 1 are arranged as shown below. Find the distance from the point A to the point F.



6. Find the area of a regular octagon inscribed in a square with side length 1.



7. In the circle below, both circles are centered at the point O. Moreover, the line segment AB is tangent to the smaller circle and has length 20 centimeters. What is the area between the two circles?



8. Solve the system

$$\begin{aligned}
\frac{1}{x} + \frac{2}{y} - \frac{1}{z} &= 3 \\
\frac{3}{x} - \frac{4}{y} + \frac{2}{z} &= -1 \\
\frac{2}{x} + \frac{2}{y} - \frac{2}{z} &= 5
\end{aligned}$$

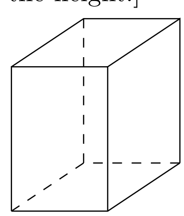
9. Simplify $\dfrac{(a+b)^4 - (a-b)^4}{8ab}$.

10. The graphs $y = x + 2$ and $y = ax^2$ intersect at a point A whose x-coordinate is 3. Find the value of a.

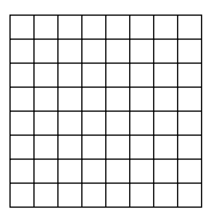
11. The roots of $x^3 + 4x^2 - 7x - 10 = 0$ are -5, -1, and 2. What are the roots of $(x-3)^3 + 4(x-3)^2 - 7(x-3) - 10 = 0$?

12. Let x and y be two distinct positive real numbers whose arithmetic mean is 13 and whose geometric mean is 12. The arithmetic mean of two numbers is $\frac{x+y}{2}$ and the geometric mean of two numbers is $\sqrt{xy}$. Find x and y.

13. Let B be a rectangular box with a square base. If the surface area of the box is 56in^2 and the girth is 14in, find the dimensions of the box. [The girth is the perimeter of the base plus the height.]



14. Two rectangles are considered different if they have either different dimensions or a different location. How many different rectangles consisting of an integral number of squares can be drawn on an 8×8 chessboard?



15. How many positive integers less than 1000 are divisible neither by 5 nor by 7?

2. Solutions

1. L has length $\sqrt{2}$. The region S is a rectangle of dimensions $2 \times \sqrt{2}$ and two semicircles of radius 1. So the area of S is $2(\pi/2) + 2\sqrt{2} = \pi + 2\sqrt{2}$.
2. The area is the area of two quarter circles minus the area of the square. This is $2\pi 4^2/4 - 16 = 8\pi - 16$.
3. Let (a, b) be the center of the circle and r its radiu. An equation for the circle is $(x-a)^2 + (y-b)^2 = r^2$. Since the circle is tangent to both the x and y axes then $(r, 0)$ and $(0, r)$ belong to the circle. This implies that $r = a = b$. Since $(9, 8)$ also belongs to the circle, then

$$(9-r)^2 + (8-r)^2 = r^2$$
$$81 - 18r + r^2 + 64 - 16r + r^2 = r^2$$
$$r^2 - 34r + 145 = 0$$
$$(r-5)(r-29) = 0$$

   So $r = 29$.
4. Let r be the radius of the smaller circle. From the picture we have the right triangle with sides 1, $1-r$, and $1+r$. So $(1+r)^2 = 1^2 + (1-r)^2$. Thus $4r = 1$ and $r = \frac{1}{4}$.
5. The height of an equilateral triangle is $\sqrt{3}/2$. To get from A to F we move three heights to the right and up half a length. Thus we can draw a right triangle with legs of length $3\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$. The distance AF is the length of the hypotenuse which is $\sqrt{7}$.
6. Let x be the length of a side of the octagon. Then $x + 2\frac{x}{\sqrt{2}} = 1$, or $x = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$. Since the area of the octagon is a unit square minus four corners which fit together to form a small square with side x, the area is $1 - (\sqrt{2}-1)^2 = 2\sqrt{2} - 2 \cong .828427124$.
7. Let r and R be the radii. From the picture we see that $r^2 + 10^2 = R^2$. Thus $\pi R^2 - \pi r^2 = 100\pi$.
8. Let $u = 1/x$, $v = 1/y$ and $w = 1/z$. 2 Row 1 - Row 3 tells us that $2v = 1$ or that $v = 1/2$. Using this, Row 1 and Row 2 simplify to the equations

$$u - w = 2$$
$$3u + 2w = 1$$

   Twice the 1st plus the second gives $5u = 5$ or $u = 1$ and thus $w = -1$. Converting back we find $(x, y, z) = (1, 2, -1)$.
9. Expanding we get $\frac{(a^4+4a^3b+6a^2b^2+4ab^3+b^4)-(a^4-4a^3b+6a^2b^2-4ab^3+b^4)}{8ab} = \frac{8a^3b+8ab^3}{8ab} = a^2 + b^2$
10. Since the x-coordinate of A is 3, then its y-coordinate is $y = 3 + 2 = 5$. So $A = (3, 5)$ and belongs to the graph of $y = ax^2$. Hence $5 = a(3)^2$, i.e. $a = 5/9$.
11. $x - 3 = -5, -1$, or 2, so $x = -2, 2$, or 5.
12. Start with the equations $x + y = 2(13)$ and $xy = 12^2$. Solving for y and substituting yields $x^2 - 26x + 144 = 0$. The quadratic formula gives 8 and 18 as a solution.
13. Start with the equations $2x^2 + 4xz = 56$ and $4x + z = 14$. Solving for z and substituting yields $14x^2 - 56x + 56 = 0$. This factors as $14(x-2)^2 = 0$ so $x = 2$ and $z = 6$.

14. $\binom{9}{2}^2 = 1296$. This can also be done much more explicitly without binomial coefficients at this level.
15. There are 199 numbers less than 1000 which are divisible by 5, 142 which are divisible by 7, and 28 which are divisible by both. Thus $999 - 199 - 142 + 28 = 686$ are not divisible by either.