EF Exam

TAMU High School Mathematics Contest
November 18, 2000

1. Find the area of the triangle with vertices $(2,1)$, $(3,7)$ and $(1,6)$.

2. Find the cosine of the acute angle of intersection of the lines given by $y = -2x + 3$ and $y = 4x - 6$.

3. Find all solutions to the system of equations

$$\begin{aligned} \cos(x+y) - \cos(x) - \cos(y) &= 1 \\ \sin(x+y) - \sin(x) - \sin(y) &= 0 \end{aligned}$$

   for x and y in the interval $[0, 2\pi]$.

4. How many solutions are there to the equation

$$\cos^3(x) - 3\cos(x)\sin^2(x) = \cos(3x)$$

   for x between 0 and 1?

5. Find the solution to

$$\tan(\arcsin(x)) = 3$$

   with $0 < x < 1$.

6. Let $f(x)$ be a differentiable function and suppose a is a real number such that $f(a) > 0$ and $f'(a) > 0$. Let P be the point $(a, f(a))$, let Q be the point $(a, 0)$ and let R be the point where the tangent line to the graph of $f(x)$ at P intersects the x-axis. Give a formula for the area of the triangle with vertices PQR.

7. Suppose the xy plane represents a level floor with the units along each axis measured in inches. A ball of radius 3 inches is sitting at the origin, and the ball is suddenly rolled along the line $y = 3x$ into the first quadrant until it strikes a vertical wall that is placed along the line $y = 20 - x$. Give the x-coordinate of the point on the line which lies directly below the point of contact of the ball with the wall when the ball strikes the wall.

8. Suppose f is a positive continuous function on the interval $[-2, 3]$ and $A(t)$ is the area of the region bounded by the graph of $y = f(x)$ and the lines $y = 0$, $x = -2$ and $x = t$ for t between -2 and 3. Compute

$$\lim_{t \to 3^-} \frac{A(3) - A(t)}{3 - t}.$$

9. A particle sits at the point in the xy plane and awaits instructions. Every second, it is told to take its current coordinates (x, y) and use them to move to a new point given by

$$\left(3 - \frac{1}{3}x + \frac{1}{4}y, 2 + \frac{1}{2}x - \frac{1}{4}y\right)$$

Where should the particle sit initial if it wants to move as little as possible during the first 100 seconds?

10. Two circles are placed in the xy-plane. One of the circles has center $(1, 1)$ and radius 1. The other circle has center $(0, -1)$ and is tangent to the circle centered at $(1, 1)$. What are the possible radii of the circle centered at $(0, -1)$?

11. For each natural number n, a number θ_n is chosen at random between 0 and 2π, and a circular sector C_n of radius 1 and central angle θ_n radians is created. Then a sequence $\{A_k\}_{k=1}^{\infty}$ is created by defining

$$A_k = \frac{1}{k}\sum_{i=1}^{k} area\,(C_i)$$

where $area\,(C_i)$ denotes the area of sector C_i. What is the most likely value that A_k will get close to when k gets large?

12. A particle starts at the point $(2, -1)$ and moves 3 units in the direction of the point $(2, 3)$. Afterwards, it is rotated $30°$ counter-clockwise about the point $(-1, -2)$. What is the final position of the particle?

13. Four points are placed in the plane and denoted $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$, $P_4 = (x_4, y_4)$ so that the quadrilateral $P_1P_2P_3P_4$ is convex. Denote the triangles $T_1 = \Delta P_2P_3P_4$, $T_2 = \Delta P_1P_3P_4$, $T_3 = \Delta P_1P_2P_4$, $T_4 = \Delta P_1P_2P_3$. Let Q_1, Q_2, Q_3, Q_4 denote the centroids of triangles T_1, T_2, T_3, T_4 respectively and form the lines L_i that pass through the points P_i and Q_i for $i = 1, 2, 3, 4$. These lines intersect in a common point. Find this common point of intersection.

14. Suppose that circles of equal diameter are packed tightly in 4 rows inside an equilateral triangle with side length 10. What is the reciprocal of the radius of each of the circles?

15. Suppose that

$$f(n) = \log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \cdots \cdot \log_{n-1} n$$

Find

$$\sum_{k=2}^{9} f\left(2^k\right)$$

16. Let f be a polynomial function of degree n such that

$$f(k) = \frac{k}{k+1}$$

for all $k = 0, 1, ..., n$. Find $f(n+1)$.

17. A class of 100 students were scheduled to take an exam. The instructor created three versions of the exam and distributed them to the students. The class average on the exam was 75.62. Fewer than 30 students took version 1 and fewer than 30 students took version 2. If the average on version 1 was 80, the average on version 2 was 82 and the average on version 3 was 70, then how many students took version 1 of the exam?