FALL 2002
HIGH SCHOOL MATH CONTEST
CD EXAM SOLUTIONS

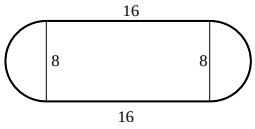
1. If
$$a + b = 21$$
$$2a - \frac{b}{2} = 7\ ,$$
then
$$2a + 2b = 42$$
$$2a - \frac{b}{2} = 7\ .$$
Thus $\dfrac{5b}{2} = 35 \Rightarrow b = 14 \Rightarrow a = 21 - b = 21 - 14 = 7 \Rightarrow a + 2b = 7 + 2(14) = 35$.

2. Let x be the original price of the suit. Then the sale price is $x - (20/100)x = (4/5)x$ and the clearance price is $(85/100)(4/5)x = (17/25)x = \$170$. Thus, $x = \$250$.

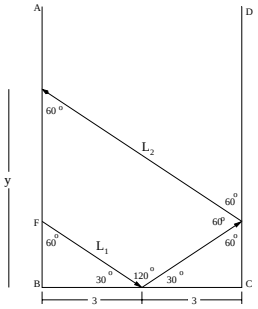
3. Since $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$, the answer is $n = 8$.

4. After t hours, Cornelius is $2 + 15t$ miles due east of Mephisto's starting position and Mephisto has traveled due east a distance of $20t$ miles. Mephisto passes Cornelius when
$$20t = 2 + 15t$$
$$5t = 2$$
$$t = \frac{2}{5} \text{ hrs } = 24 \text{ minutes}$$

5. The length of the stretched band is $2(16) + 2(4\pi) = 32 + 8\pi$.



6. L_1 and L_2 are parallel. The length of segment FB is $3\tan 30^\circ = \sqrt{3}$. L_2 passes through the point $(6, \sqrt{3})$ and has slope $= -1/\sqrt{3}$. The equation of L_2 is $y = \sqrt{3} - (1/\sqrt{3})(x - 6)$ and when $x = 0$ we have that $y = \sqrt{3} - (1/\sqrt{3})(6) = 3\sqrt{3}$.



7. The area of a square with side x is x^2. The area of a circle with radius r is πr^2. These areas are equal, so $x^2 = \pi r^2$, giving $x/r = \sqrt{\pi}$. The ratio of the perimeter of the square to the circumference of the circle is

$$\frac{4x}{2\pi r} = \frac{2}{\pi}\left(\frac{x}{r}\right) = \frac{2}{\pi}\sqrt{\pi} = \frac{2}{\sqrt{\pi}} .$$

8. $\left(1-\frac{1}{2}\right)\times\left(1-\frac{1}{3}\right)\times\left(1-\frac{1}{4}\right)\times\cdots\times\left(1-\frac{1}{8986}\right)\times\left(1-\frac{1}{8987}\right) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{8985}{8986}\right)\left(\frac{8986}{8987}\right) = \frac{1}{8987}$.

9. The area of each quarter circle is $(1/4)\pi(4)^2 = 4\pi$. The area of square $ABCD$ is 16. The area of the intersection is $4\pi + 4\pi - 16 = 8\pi - 16$.

10. Let $u = x + (1/x)$. Then

$$\begin{aligned} 2u^2 - u &= 10 \\ 2u^2 - u - 10 &= 0 \\ (2u-5)(u+2) &= 0 , \end{aligned}$$

so $u = -2$ and $u = 5/2$.

If $u = -2$, then

$$\begin{aligned} x + \frac{1}{x} &= -2 \\ x^2 + 2x + 1 &= 0 \\ (x+1)^2 &= 0 \\ x &= -1 . \end{aligned}$$

If $u = 5/2$, then

$$\begin{aligned} x + \frac{1}{x} &= \frac{5}{2} \\ 2x^2 - 5x + 2 &= 0 \\ (2x-1)(x-2) &= 0 \\ x &= \frac{1}{2} , \ x = 2 . \end{aligned}$$

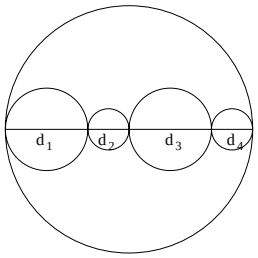
The solutions are $x = -1$, $1/2$, 2.

11. Let x and $x+2$ be the consecutive positive even integers. Then

$$\begin{aligned} (x+2)^2 - x^2 &= 1900 \\ x^2 + 4x + 4 - x^2 &= 1900 \\ 4x + 4 &= 1900 \\ 4x &= 1896 \\ x &= 474 . \end{aligned}$$

The smaller of these two integers is 474.

12. The sum of the circumferences of the inner circles is $\pi d_1+\pi d_2+\pi d_3+\pi d_4 = \pi(d_1+d_2+d_3+d_4) = 20\pi$, which is the circumference of the big circle. The total circumference of all five circles is $20\pi+20\pi = 40\pi$.



13. Since $x^2+ax+(b+2)=0$ has real roots, the discriminant is nonnegative. Thus,

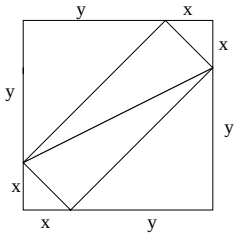
$$a^2-4(1)(b+2)\geq 0$$
$$a^2+b^2\geq b^2+4(b+2)=(b+2)^2+4\geq 4\ .$$

When $a=0$ and $b=-2$, then $a^2+b^2=4$. Thus, the least possible value for a^2+b^2 is 4.

14. The area removed is x^2+y^2. The length of a diagonal is

$$\sqrt{(\sqrt{2}x)^2+(\sqrt{2}y)^2}=\sqrt{2}\sqrt{x^2+y^2}=10\sqrt{6}\ .$$

Thus, $x^2+y^2=300$.



15. The sum of the angles of a triangle is 180°. Adding the equations

$$A+C+e=180$$
$$B+D+a=180$$
$$C+E+b=180$$
$$A+D+c=180$$
$$B+E+d=180$$

gives

$$2(A+B+C+D+E)+(a+b+c+d+e)=5(180)$$
$$2(A+B+C+D+E)+540=900$$
$$2(A+B+C+D+E)=360$$
$$A+B+C+D+E=180\ .$$