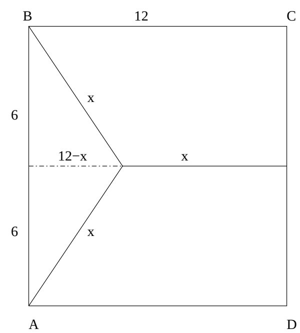
FALL 2002
HIGH SCHOOL MATH CONTEST
DE EXAM SOLUTIONS

1. Using the Pythagorean Theorem,

$$\begin{aligned} x^2 &= (12-x)^2 + 6^2 \\ x^2 &= 144 - 24x + x^2 + 36 \\ 24x &= 180 \\ x &= \frac{180}{24} = \frac{15}{2} \ . \end{aligned}$$



2. If $g(x) = f^{-1}(x)$, then

$$\begin{aligned} f(g(x)) &= x \\ 2g(x) - 6 &= x \\ g(x) &= \frac{x+6}{2} \ , \end{aligned}$$

and

$$\begin{aligned} f(x) &= g(x) \\ 2x - 6 &= \frac{x+6}{2} \\ 4x - 12 &= x + 6 \\ 3x &= 18 \\ x &= 6 \ . \end{aligned}$$

3. If $x = 1/(4-y)$, then

$$\begin{aligned} \frac{1}{x} &= 4 - y \\ 4x - yx &= 1 \end{aligned}$$

and

$$\frac{1}{x} + 4x + y - yx - 1 = \frac{1}{x} + (4x - yx) + y - 1 = (4-y) + 1 + y - 1 = 4 \ .$$

4. We have the equations

$$\begin{aligned} x\left(1 + \frac{y}{100}\right) &= 30 \\ y\left(1 + \frac{x}{100}\right) &= 25 \ , \end{aligned}$$

giving

$$x + \frac{xy}{100} = 30$$
$$y + \frac{xy}{100} = 25 \ .$$

Subtracting gives $y = x - 5$, whence

$$\begin{aligned} x - 5 + \frac{x(x-5)}{100} &= 25 \\ 100x - 500 + x^2 - 5x &= 2500 \\ x^2 + 95x - 3000 &= 0 \\ (x-25)(x+120) &= 0 \ . \end{aligned}$$

Now x must be positive, so $x = 25$.

5.

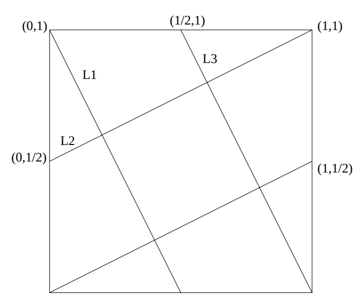
$$\begin{array}{r l} & x^2 + 2x + 6 \\ x - 3 & \overline{\big) x^3 - x^2 + 5k - 2} \\ & \underline{x^3 - 3x^2} \\ & \quad 2x^2 + 5k - 2 \\ & \quad \underline{2x^2 - 6x} \\ & \qquad 6x + 5k - 2 \\ & \qquad \underline{6x - 18} \\ & \qquad\quad 5k + 16 \end{array}$$

The remainder is $5k + 16$, so

$$\begin{aligned} 5k + 16 &= k \\ 4k &= -16 \\ k &= -4 \ . \end{aligned}$$

6. The equation of line $L1$ is $y = -2x + 1$; the equation of line $L2$ is $y = (1/2)x + (1/2)$ and the equation of line $L3$ is $y = -2x + 2$. Line $L1$ and $L2$ intersect at the point $(1/5, 3/5)$ and the lines $L2$ and $L3$ intersect at the point $(3/5, 4/5)$. The area of the center square is

$$d^2 = \text{dist}((1/5,3/5),(3/5,4/5)) = \left(\frac{3}{5} - \frac{1}{5}\right)^2 + \left(\frac{4}{5} - \frac{3}{5}\right)^2 = \frac{4}{25} + \frac{1}{25} = \frac{1}{5} \ .$$



7. If r is a fixed point of $f(x) = x^2 + ax + b$ then

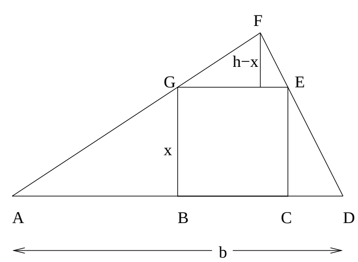
$$\begin{aligned} f(r) &= r \\ r^2 + ar + b &= r \\ r^2 + (a-1)r + b &= 0 \ , \end{aligned}$$

and there is a single real solution r when

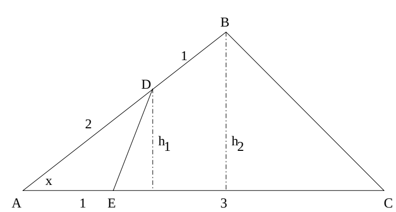
$$\begin{aligned} (a-1)^2 - 4(1)(b) &= 0 \\ b &= \frac{1}{4}(a-1)^2 \ . \end{aligned}$$

8. The sum of the areas of ΔABG and ΔCDE is $(1/2)x(b-x)$. The area of square $BCEG = x^2$; the area of $\Delta GEF = (1/2)x(h-x)$ and the area of $\Delta ADF = (1/2)bh$. Thus

$$\begin{aligned} \frac{1}{2}bh &= \frac{1}{2}x(b-x) + x^2 + \frac{1}{2}x(h-x) \\ bh &= x(b-x) + 2x^2 + x(h-x) \\ bh &= bx - x^2 + 2x^2 + xh - x^2 \\ bh &= x(b+h) \\ x &= \frac{bh}{b+h} \ . \end{aligned}$$



9. The area of $\Delta AED = (1/2)(1)h_1 = (1/2)(1)(2\sin x) = \sin x$ and the area of $\Delta ABC = (1/2)(4)h_2 = (1/2)(4)(3\sin x) = 6\sin x$. Thus, the ratio of the area of ΔAED to the area of ΔABC is $1/6$.



10. If $a^{2x} = 3$, then $a^x = \sqrt{3}$ and

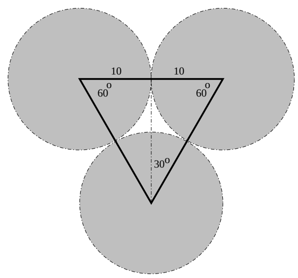
$$\frac{a^{3x} + a^{-3x}}{a^x + a^{-x}} = \frac{(a^x)^3 + (a^{-x})^3}{a^x + a^{-x}} = \frac{(\sqrt{3})^3 + \left(\frac{1}{\sqrt{3}}\right)^3}{\sqrt{3} + \frac{1}{\sqrt{3}}} = \frac{3\sqrt{3} + \frac{1}{3\sqrt{3}}}{\sqrt{3} + \frac{1}{\sqrt{3}}} = \frac{3\sqrt{3} + \frac{1}{3\sqrt{3}}}{\sqrt{3} + \frac{1}{\sqrt{3}}} \cdot \frac{3\sqrt{3}}{3\sqrt{3}} = \frac{28}{12} = \frac{7}{3} \ .$$

11. There are integers j, k, and l such that

$$n = 3k + 2$$
$$n = 5j + 3$$
$$n = 7l + 2 \ .$$

The smallest possible value of $n = 23$ and this corresponds to $k = 7$, $j = 4$, and $l = 3$.

12. The area of the equilateral triangle is $(1/2)(20)(10\tan(\pi/3)) = 100\sqrt{3}$. The area of the shaded region within the equilateral triangle is $(180/360)\cdot\pi(10)^2 = 50\pi$. The area of the unshaded region between the circles is $100\sqrt{3} - 50\pi$.



13.

$$5\tan\theta = 6\cos\theta$$
$$5\left(\frac{\sin\theta}{\cos\theta}\right) = 6\cos\theta$$
$$5\sin\theta = 6\cos^2\theta$$
$$5\sin\theta = 6(1-\sin^2\theta)$$
$$6\sin^2\theta + 5\sin\theta - 6 = 0$$
$$(3\sin\theta - 2)(2\sin\theta + 3) = 0 \ ,$$

giving $\sin\theta = 2/3$ and $\sin\theta = -3/2$ (this is impossible!). Thus, $\sin\theta = 2/3$.

14. The multiples of 4 are 25 in number and are

$$\{4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96, 100\} \ .$$

The multiples of 9 that are not also multiples of 4 are 9 in number and are

$$\{9, 18, 27, 45, 54, 63, 81, 90, 99\} \ .$$

The probability that a number chosen at random is neither a multiple of 4 nor 9 is $(100-(25+9))/100 = 66/100 = 33/50$.

15. Let F, S and T be the prize money for first, second and third, respectively. Then

$$F = 50 + S$$
$$S = 40 + T \ ,$$

giving

$$\begin{aligned} F + S + T &= 520 \\ F + (F-50) + (F-90) &= 520 \\ 3F - 140 &= 520 \\ 3F &= 660 \\ F &= \$220 \ . \end{aligned}$$

16. Let $\theta = 2\arcsin(1/3)$, then $\sin(\theta/2) = 1/3$ and $\cos(\theta/2) = (2\sqrt{2})/3$. Thus

$$\sin\theta = \sin 2(\theta/2) = 2\sin(\theta/2)\cos(\theta/2) = 2(1/3)((2\sqrt{2})/3) = \frac{4\sqrt{2}}{9} \ .$$

17. Now

$$\begin{aligned} \text{abcd } \times 1999 &= \underline{\quad\quad}1111 \\ (\text{abcd } \times 1999) + \text{abcd} &= \underline{\quad\quad}1111 + \text{abcd} \\ 2000(\text{abcd}) &= \underline{\quad\quad}1111 + \text{abcd} \\ \underline{\quad\quad}(2\text{d} \bmod 10)000 &= \underline{\quad\quad}1111 + \text{abcd} \ , \end{aligned}$$

Thus $d = 9$, $c = 8$, $b = 8$ and $1 + 1 + a = 2\text{d} \bmod 10 = 18 \bmod 10 = 8$. Thus, $a = 6$ and the number is 6889 .