EF Exam

1. The tangent line to the graph of the equation $y = x^2 + 7x + 11$ at the point $(3, 41)$ crosses the y-axis at the point $(0, a)$, where a is a real number. What is the value of a?

2. Give the units digit of the following sum: $1! + 2! + 3! + \ldots + 99!$

3. The graphs of the equations $y = 5 - |x - 2|$ and $y = |x + 2| - 5$ enclose a rectangular region in the coordinate plane. What is the area of this rectangle?

4. Let f be the function given by $f(x) = 3x + 4$ for all real x. Compute the sum

$$S = f(0) + f(1) + f(2) + ... + f(100).$$

5. Suppose that the greatest possible value of $2 \sin x + 3 \cos x$ (for $x \in [0, 2\pi)$) is M, and a is a real number such that $2 \sin a + 3 \cos a = M$. (So the quantity $2 \sin x + 3 \cos x$ is maximized at $x = a$.) Compute $\tan a$.

6. Suppose that square $ABCD$ is drawn in the plane. Two circles are then drawn: a circle inscribed in square $ABCD$, and a circle circumscribed about square $ABCD$. Compute the ratio of the area of the larger circle to the area of the smaller circle.

7. Suppose that f and g are differentiable functions such that $f(0) = 3$, $f'(0) = 2$, $g(0) = 7$, and $g'(0) = -1$. Assuming that the function h given by $h(x) = f(x)/g(x)$ is well-defined and differentiable everywhere, what is the value of $h'(0)$?

8. If a, b, and c are positive integers such that $a^2 + b^2 + c^2 = 101$, then what is the greatest possible value of $a + b + c$?

9. Let S be the set of integers which are divisible by 5, and let T be the set of integers which are divisible by 7. How many positive integers less than 1000 are **not** in $S \cup T$, the union of S and T?

10. Let f be the function defined by $f(x) = (x^2 + 4x + 5)^2$ for all real numbers x. What is the least positive integer n such that $f'(n) > 2002$?

11. Suppose we take the line segment whose endpoints are (0, 0) and (1, 0), and rotate it 360 degrees around the point (2, 2). During this process, the line segment sweeps out a region of area A. Compute A.

12. The parabola given by the equation $y = ax^2 + bx + 1$ (where a and b are real numbers) passes through the points (-1, 1776) and (1, 1812). What is the x-coordinate of the vertex of the parabola?

13. Define a sequence $\langle a_n \rangle$ of positive numbers as follows: let $a_1 = 1$; and for all positive integers k, let $a_{2k} = 0.9(a_{2k-1})$ and $a_{2k+1} = 1.1(a_{2k})$. (Then the first few terms of the sequence are: $1, 0.9, 0.99, 0.891, 0.9801, 0.88209, \ldots$) Compute the sum of the series

$$\sum_{n=1}^{\infty} a_n.$$

14. There exists a positive number k such that

$$\log_2 x + \log_4 x + \log_8 x = \log_k x$$

for all positive real numbers x. If $k = \sqrt[b]{a}$, where a and b are positive integers, what is the smallest possible value of $a + b$?

15. Compute the following limit:

$$\lim_{h \to 0} \frac{\sin(\pi/3 + h) - \sin(\pi/3)}{\cos(\pi/3 + h) - \cos(\pi/3)}$$

16. Compute the maximum value of the function

$$f(x) = \sin^6 x \cos^6 x + \sin^4 x \cos^8 x$$

on the interval $[0, \pi/2]$, where x is in radians.

17. Suppose that x and y are complex numbers such that $x + y = xy = 1$. What is the value of $x^3 + y^3$?

18. If r is a real number, then what is the smallest possible distance from the origin $(0, 0)$ to the vertex of the parabola whose equation is $y = x^2 + rx + 1$?

19. Define a function f by

$$f(x) = \frac{x^{2002} - 1}{x - 1}$$

for all $x \neq 1$. Compute the limit $\lim_{x \to 1} f'(x)$.

20. It is an interesting (and very useful) fact that the sum of the reciprocals of the nonzero perfect squares is $\frac{\pi^2}{6}$. (That is, the sum of the series $\sum_{n=1}^{\infty} 1/n^2$ is $\frac{\pi^2}{6}$.) Using this information, compute the sum of the reciprocals of the nonzero perfect squares that are relatively prime to 6; that is, share no common divisors (other than 1 and -1) with 6.