Solutions to 2002 Texas A&M University
High School Power Team Exam

$$S = \{ax^2 + bx + c \mid a, b, c \in R\}$$

$P(x) \sim Q(x)$ if there exist numbers α, β with $\alpha \neq 0$ such that $P(x) = Q(\alpha x + \beta)$
1. Prove that $\sim$ is an equivalence relation.

(a) Using $\alpha = 1, \beta = 0$ we have $P(x) = P(1x + 0)$ so $P(x) \sim P(x)$.

(b) Suppose $P(x) \sim Q(x)$. Then there exist α, β with $\alpha \neq 0$ such that $P(x) = Q(\alpha x + \beta)$. Let $y = \alpha x + \beta$. Then $x = \alpha^{-1}(y - \beta) = \alpha^{-1}y - \alpha^{-1}\beta$. So

$$Q(y) = P(\alpha^{-1}y - \alpha^{-1}\beta) \quad \text{where} \quad \alpha^{-1} \neq 0.$$

Hence

$$Q(x) = P(\alpha^{-1}x - \alpha^{-1}\beta) \quad \text{and} \quad Q(x) \sim P(x).$$

(c) Suppose $P(x) \sim Q(x)$ and $Q(x) \sim R(x)$. Then $P(x) = Q(\alpha x + \beta)$ and $Q(x) = R(\gamma x + \delta)$ where $\alpha \neq 0$, $\gamma \neq 0$. So $P(x) = Q(y)$ and $Q(y) = R(z)$ with $y = \alpha x + \beta$ and $z = \gamma y + \delta$. Hence

$$\begin{aligned} z = \gamma y + \delta &= \gamma(\alpha x + \beta) + \delta \\ &= \gamma\alpha x + \gamma\beta + \delta \quad \text{where} \quad \gamma\alpha \neq 0. \end{aligned}$$

So $P(x) = R(z) = R(\gamma\alpha x + \gamma\beta + \delta)$ and $P(x) \sim R(x)$

$$(R(\gamma\alpha x + \gamma\beta + \delta) = R(\gamma(\alpha x + \beta) + \delta) = Q(\alpha x + \beta) = P(x)).$$

2. Let $Q(x) = ax^2 + bx + c, a \neq 0$, have two different real roots. Find a polynomial of the form $P(x) = kx^2 + x$ such that $P(x)$ is equivalent to $Q(x)$.

Transform $Q(x)$ by $\alpha x + \beta$ to obtain

$$\begin{aligned} Q(\alpha x + \beta) &= a(\alpha x + \beta)^2 + b(\alpha x + \beta) + c \\ &= a\alpha^2 x^2 + (2a\alpha\beta + \alpha b)x + a\beta^2 + b\beta + c. \end{aligned}$$

Let β be any root of $Q(x)$, and obtain

$$Q(\alpha x + \beta) = a\alpha^2 x^2 + (2a\alpha\beta + \alpha b)x.$$

Since β is not a repeated root of $Q(x)$, then $Q'(\beta) \neq 0$, i.e. $2a\beta + b \neq 0$. So let $\alpha = \frac{1}{2a\alpha + b}$ to obtain

$$Q\left(\frac{1}{2a\beta + b}x + \beta\right) = a\left(\frac{1}{2a\beta + b}\right)^2 x^2 + x.$$

Hence $Q(x) \sim kx^2 + x$ where $k = a\left(\frac{1}{2a\beta+b}\right)^2$ and β is a root of $Q(x)$.

3. (i) $Q(x)$ is equivalent to a polynomial of the form $k(x^2 + 1)$ for some nonzero constant k when $Q(x)$ has complex roots. Indicate k.

Transform $Q(x)$ by $\alpha x + \beta$ $(\alpha \neq 0)$ to obtain

$$Q(\alpha x + \beta) = a\alpha^2 x^2 + \alpha(2a\beta + b)x + a\beta^2 + b\beta + c.$$

Since $\alpha \neq 0$ let $2a\beta + b = 0$ or $\beta = -\frac{b}{2a}$ ($a \neq 0$ since $Q(x)$ has 2 roots). With $\beta = -\frac{b}{2a}$ we have

$$a\beta^2 + b\beta + c = -\frac{b^2}{4a} + c = -\frac{b^2}{4a} + \frac{4ac}{4a} = \frac{-b^2 + 4ac}{4a} > 0$$

since $b^2 - 4ac < 0$ ($Q(x)$ has complex roots). So $Q\left(\alpha x - \frac{b}{2a}\right) = a\alpha^2 x^2 + \frac{-b^2+4ac}{4a}$. We want

$$\begin{aligned} a\alpha^2 &= \frac{-b^2 + 4ac}{4a} \\ \alpha^2 &= \frac{4ac - b^2}{4a^2} \\ \alpha &= \frac{\sqrt{4ac - b^2}}{|2a|}. \end{aligned}$$

Then

$$\begin{aligned} Q\left(\frac{\sqrt{4ac - b^2}}{|2a|}x - \frac{b}{2a}\right) &= \frac{-b^2 + 4ac}{4a}x^2 + \frac{-b^2 + 4ac}{4a} \\ &= \frac{4ac - b^2}{4a}(x^2 + 1), \qquad k = \frac{4ac - b^2}{4a}. \end{aligned}$$

So $Q(x) \sim \frac{4ac-b^2}{4a}(x^2 + 1)$.

3. (ii) $Q(x)$ is equivalent to a polynomial of the form $k(x^2 - 1)$ when $Q(x)$ has two distinct real roots. Indicate k.

As in (i), let $\beta = -\frac{b}{2a}$. Then $Q\left(\alpha x - \frac{b}{2a}\right) = a\alpha^2 x^2 + \frac{-b^2+4ac}{4a}$, where $-b^2 + 4ac < 0$ since $Q(x)$ has two distinct real roots. So $Q\left(\alpha x - \frac{b}{2a}\right) = a\alpha^2 x^2 - \frac{b^2-4ac}{4a}$ where $\frac{b^2-4ac}{|4a|} > 0$. We want

$$\begin{aligned} a\alpha^2 &= \frac{b^2 - 4ac}{4a} \\ \alpha^2 &= \frac{b^2 - 4ac}{4a^2} \\ \alpha &= \frac{\sqrt{b^2 - 4ac}}{|2a|}. \end{aligned}$$

This gives

$$\begin{aligned} Q\left(\frac{\sqrt{b^2 - 4ac}}{|2a|}x - \frac{b}{2a}\right) &= \frac{b^2 - 4ac}{4a}x^2 - \frac{b^2 - 4ac}{4a} \\ &= \frac{b^2 - 4ac}{4a}(x^2 - 1), \qquad k = \frac{b^2 - 4ac}{4a}. \end{aligned}$$

So $Q(x) \sim \frac{b^2-4ac}{4a}(x^2 - 1)$.

3. (iii) $Q(x)$ is equivalent to x^2 when $Q(x)$ has a double root and $a > 0$. $Q(x)$ is equivalent to $-x^2$ when $a < 0$ and $Q(x)$ has a double root.

As before transform $Q(x) = ax^2 + bx + c$ by $\alpha x + \beta$ to obtain

$$Q(\alpha x + \beta) = a\alpha^2 x^2 + \alpha(2a\beta + b)x + a\beta^2 + b\beta + c.$$

Let β be the root of $Q(x)$. So

$$a\beta^2 + b\beta + c = 0$$
$$\text{and} \quad 2a\beta + b = 0 \qquad \left(\beta = -\frac{b}{2a}\right)$$

since β is a repeated root. Then $Q(\alpha x + \beta) = a\alpha^2 x^2$. If $a > 0$ then let $a\alpha^2 = 1$ and $\alpha = \frac{1}{\sqrt{a}}$. This gives

$$Q\left(\frac{1}{\sqrt{a}}x - \frac{b}{2a}\right) = x^2.$$

If $a < 0$ then $a\alpha^2 = 1$ is not possible, let

$$\alpha = \frac{1}{\sqrt{-a}}$$

to obtain

$$Q\left(\frac{1}{\sqrt{-a}}x - \frac{b}{2a}\right) = -x^2$$

(iv) $Q(x)$ is equivalent to x if $Q(x)$ has degree 1

$$Q(x) = bx + c.$$

Transform $Q(x)$ by $\alpha x + \beta$ to obtain

$$\begin{aligned} Q(\alpha x + \beta) &= b(\alpha x + \beta) + c \\ &= \alpha bx + b\beta + c. \end{aligned}$$

Let $\beta = \frac{-c}{b}$. Then $b\beta + c = 0$. Let $\alpha = \frac{1}{b}$. Then $\alpha b = 1$. So

$$Q\left(\frac{1}{b}x - \frac{c}{b}\right) = x.$$

(v) $Q(x) \sim k$ when $Q(x) = k$.

Trivial.

4. Let $Q(x) = ax^2 + bx + c$ with $a \neq 0$. Let $\Delta = b^2 - 4ac$. Let $\varepsilon > 0$. Prove there exists a polynomial $P(x)$ with discriminant $\varepsilon\Delta$ and $Q(x) \sim P(x)$.

Transform $Q(x)$ by $\alpha x + \beta$ to obtain

$$\begin{aligned} Q(\alpha x + \beta) &= a(\alpha x + \beta)^2 + b(\alpha x + \beta) + c \\ &= a\alpha^2 x^2 + (2a\alpha\beta + b\alpha)x + a\beta^2 + b\beta + c. \end{aligned}$$

The discriminant of $Q(\alpha x + \beta)$ is

$$\begin{aligned} &(2a\alpha\beta + b\alpha)^2 - 4(a\alpha^2)(a\beta^2 + b\beta + c) \\ &\quad = 4a^2\alpha^2\beta^2 + 4ab\alpha^2\beta - 4a^2\alpha^2\beta^2 - 4ab\beta\alpha^2 - 4ac\alpha^2 + b^2\alpha^2 \\ &\quad = b^2\alpha^2 - 4ac\alpha^2 = (b^2 - 4ac)\alpha^2 \\ &\quad = \Delta\alpha^2. \end{aligned}$$

We want $\Delta\alpha^2 = \varepsilon\Delta$, so let $\alpha = \sqrt{\varepsilon}$. Then for any β, $Q(\sqrt{\varepsilon}\, x + \beta) = P(x)$ has discriminant $\varepsilon\Delta$ and

$$Q(x) \sim P(x)$$

$$S = \{ax^2 + bx + c \mid a, b, c \in R\}$$

$P(x) \approx Q(x)$ if there exist numbers $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma \neq 0$ such that

$$P(x) = (\gamma x + \delta)^2 Q\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right).$$

(5) Prove that $\approx$ is an equivalence relation.

(a) Using $a = 1, \beta = 0, \gamma = 0, \delta = 1$ gives

$$P(x) = (0x + 1)^2 P\left(\frac{1x + 0}{0x + 1}\right) \Rightarrow P(x) \approx P(x).$$

(b) Suppose $P(x) \approx Q(x)$. Then

$$P(x) = (\gamma x + \delta)^2 Q\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) \text{ where } \alpha\delta - \beta\gamma \neq 0.$$

Let $y = \frac{\alpha x + \beta}{\gamma x + \delta}$. Solving for x in terms of y gives

$$x = \frac{\beta - \delta y}{\gamma y - \alpha} = \frac{-\delta y + \beta}{\gamma y - \alpha} \text{ where } (-\alpha)(-\delta) - \beta\gamma \neq 0.$$

Also

$$\frac{y}{\alpha x + \beta} = \frac{1}{\gamma x + \delta} \text{ so } \frac{y^2}{(\alpha x + \beta)^2} = \frac{1}{(\gamma x + \delta)^2}$$

and

$$\alpha x + \beta = \alpha\left(\frac{\delta y - \beta}{\alpha - \gamma y}\right) + \beta\left(\frac{\alpha - \gamma y}{\alpha - \gamma y}\right) = \frac{y(\alpha\delta - \beta\gamma)}{\alpha - \gamma y}.$$

So

$$\begin{aligned} \frac{1}{(\gamma x + \delta)^2} = \frac{y^2}{(\alpha x + \beta)^2} &= \left(\frac{\alpha - \gamma y}{\alpha\delta - \beta\gamma}\right)^2 \\ &= \left(-\frac{\gamma}{\alpha\delta - \beta\gamma}y + \frac{\alpha}{\alpha\delta - \beta\gamma}\right)^2. \end{aligned}$$

We have

$$\frac{1}{(\gamma x + \delta)^2} P(x) = Q\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)$$

$$\left(-\frac{\gamma}{\alpha\delta - \beta\gamma}y + \frac{\alpha}{\alpha\delta - \beta\gamma}\right)^2 P\left(\frac{-\delta y + \beta}{\gamma y - \alpha}\right) = Q(y)$$

$$\left(-\frac{\gamma}{\alpha\delta - \beta\gamma}y + \frac{\alpha}{\alpha\delta - \beta\gamma}\right)^2 P\left(\frac{-\frac{\delta}{\alpha\delta - \beta\gamma}y + \frac{\beta}{\alpha\delta - \beta\gamma}}{\frac{\gamma}{\alpha\delta - \beta\gamma}y - \frac{\alpha}{\alpha\delta - \beta\gamma}}\right) = Q(y).$$

So $Q(x) \approx P(x)$.

(c) Suppose $P(x) \approx Q(x) + Q(x) \approx R(x)$. Then

$$P(x) = (\gamma_1 x + \delta_1)^2 Q\left(\frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}\right), \qquad \alpha_1\delta_1 - \beta_1\gamma_1 \neq 0$$
$$Q(x) = (\gamma_2 x + \delta_2)^2 R\left(\frac{\alpha_2 x + \beta_2}{\gamma_2 x + \delta_2}\right), \qquad \alpha_2\delta_2 - \beta_2\gamma_2 \neq 0.$$

We have

$$Q(y) = (\gamma_2 y + \delta_2)^2 R\left(\frac{\alpha_2 y + \beta_2}{\gamma_2 y + \delta_2}\right).$$

Let $y = \frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_2}$ to obtain

$$P(x) = (\gamma_1 x + \delta_1)^2 Q\left(\frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_2}\right).$$

So

$$\begin{aligned}\frac{\alpha_2 y + \beta_2}{\gamma_2 y + \delta_2} &= \frac{\alpha_2\left(\frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}\right) + \beta_2}{\gamma_2\left(\frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}\right) + \beta_2} \\ &= \frac{(\alpha_2\alpha_1 + \beta_2\gamma_1)x + (\alpha_2\beta_1 + \beta_2\delta_1)}{(\gamma_2\alpha_1 + \delta_2\gamma_1)x + (\gamma_2\beta_1 + \delta_2\delta_1)}\end{aligned}$$

where $(\alpha_2\alpha_1 + \beta_2\gamma_1)(\gamma_2\beta_1 + \delta_2\delta_1) - (\gamma_2\alpha_1 + \delta_2\gamma_1)(\alpha_2\beta_1 + \beta_2\delta_1)$

$$\begin{aligned}&= \alpha_2\alpha_1\gamma_2\beta_1 + \alpha_2\alpha_1\delta_2\delta_1 + \beta_2\gamma_1\gamma_2\beta_1 + \beta_2\gamma_1\delta_2\delta_1 \\ &\quad - \gamma_2\alpha_1\alpha_2\beta_1 - \gamma_2\alpha_1\beta_2\delta_1 - \delta_2\gamma_1\alpha_2\beta_1 - \delta_2\gamma_2\beta_2\delta_1 \\ &= (\alpha_1\delta_1 - \beta_1\gamma_1)(\alpha_2\delta_2) + (\beta_1\gamma_1 - \alpha_1\delta_1)\beta_2\gamma_2 \\ &= (\alpha_1\delta_1 - \beta_1\gamma_1)(\alpha_2\delta_2 - \beta_2\gamma_2) \neq 0.\end{aligned}$$

We now have

$$\begin{aligned}Q(y) &= (\gamma_2 y + \delta_2)^2 R\left(\frac{\alpha_2 y + \beta_2}{\gamma_2 y + \delta_2}\right) \\ Q\left(\frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}\right) &= \left(\gamma_2\frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1} + \delta_2\right)^2 R\left(\frac{(\alpha_2\alpha_1 + \beta_2\gamma_1)x + (\alpha_2\beta_1 + \beta_2\delta_1)}{(\gamma_2\alpha_1 + \delta_2\gamma_1)x + (\gamma_2\beta_1 + \delta_2\delta_1)}\right) \\ &= \left(\frac{(\gamma_2\gamma_1 + \delta_2\gamma_1)x + (\gamma_2\beta_1 + \delta_2\delta_1)}{\gamma_1 x + \delta_1}\right)^2 R\left(\frac{(\alpha_2\alpha_1 + \beta_2\gamma_1)x + (\alpha_2\beta_1 + \beta_2\delta_1)}{(\gamma_2\alpha_1 + \delta_2\gamma_1)x + (\gamma_2\beta_1 + \delta_2\delta_1)}\right).\end{aligned}$$

Now multiply both sides of the equation by $(\gamma_1 x + \delta_1)^2$ to obtain

$$\begin{aligned}P(x) &= (\gamma_1 x + \delta_1)^2 Q\left(\frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}\right) \\ &= \big((\gamma_1\alpha_1 + \delta_2\gamma_1)x + (\gamma_2\beta_1 + \delta_2\delta_1)\big)^2 R\left(\frac{(\alpha_2\alpha_1 + \beta_2\gamma_1)x + (\alpha_2\beta_1 + \beta_2\delta_1)}{(\gamma_2\alpha_1 + \delta_2\gamma_1)x + (\gamma_2\beta_1 + \delta_2\delta_1)}\right).\end{aligned}$$

So $P(x) \approx R(x)$.

6. Show that $Q(x) = x^2 - 1$ is equivalent to $P(x) = 1 - 2x$.

$$x^2 Q\left(\frac{1-x}{x}\right) = x^2\left(\left(\frac{1-x}{x}\right)^2 - 1\right)$$
$$= x^2\left(\frac{1 - 2x + x^2 - x^2}{x^2}\right) = 1 - 2x.$$

So $Q(x) \approx 1 - 2x$.

7. Which quadratic polynomials $Q(x) = ax^2 + bx + c$, $a \neq 0$, are equivalent to a polynomial of degree less than 2?

Transform $Q(x) = ax^2 + bx + c$ to obtain

$$\begin{aligned} P(x) &= (\gamma x + \delta)^2 Q\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) \\ &= a(\alpha x + \beta)^2 + b(\alpha x + \beta)(\gamma x + \delta) + c(\gamma x + \delta)^2 \\ &= (a\alpha^2 + b\alpha\gamma + c\gamma^2)x^2 + (2a\alpha\beta + b\alpha\delta + b\beta\gamma + 2c\gamma\delta)x \\ &\quad + a\beta^2 + b\beta\delta + c\delta^2. \end{aligned}$$

For $\deg P(x) < 2$ we need $a\alpha^2 + b\alpha\gamma + c\gamma^2 = 0$. Since $\alpha\delta - \beta\gamma \neq 0$ then at least one of α, γ is nonzero. Assume $\alpha \neq 0$ and divide by α^2 to obtain

$$a + b\frac{\gamma}{\alpha} + c\left(\frac{\gamma}{\alpha}\right)^2 = 0.$$

So $\frac{\gamma}{\alpha}$ is a real root of $Q(x)$.

Now suppose r is a real root of $Q(x)$. Let $\gamma = r$ and $\alpha = 1$. Transforming $Q(x)$ gives

$$\underbrace{(a + br + cr^2)}_{0}x^2 + (2a\beta + b\delta + b\beta r + 2cr\delta)x + a\beta^2 + b\beta\delta + c\delta^2$$

So $Q(x)$ is equivalent to a polynomial of degree < 2 if and only if $Q(x)$ has a real root.

8. Describe, in a useful way, each equivalence class of $\approx$, and give a representative for each class.

ANSWER: Let $Q(x) = ax^2 + bx + c$ and let $\Delta = b^2 - 4ac$

$Q(x) \approx x^2 + 1$	when $\Delta < 0$ and $Q(x) > 0$ for all x
$Q(x) \approx -x^2 - 1$	when $\Delta < 0$ and $Q(x) < 0$ for all x
$Q(x) \approx x$	when $\Delta > 0$
$Q(x) \approx 1$	when $\Delta = 0$ and $Q(x) \geq 0$ for all x
$Q(x) \approx -1$	when $\Delta = 0$ and $Q(x) \leq 0$ for all x
$Q(x) \approx 0$	when $Q(x) \equiv 0$

Solution to 8: Let $Q(x) = ax^2 + bx + c$ with $\Delta = b^2 - 4ac < 0$ and $Q(x) > 0$ for all x. Since $Q(x) > 0$ for all x then $a > 0$. The equivalence relation $\sim$ in problems 1–4 is a special case of $\approx$ using $\gamma = 0$, $\delta = 1$. By 3(i) $Q(x) \approx \frac{4ac-b^2}{4a}(x^2+1)$ where $k \equiv \frac{4ac-b^2}{4a} > 0$ (since $a > 0$). Now it is enough to show that $k(x^2+1) \approx x^2 + 1$ when $k > 0$. Let $\alpha = \gamma = \frac{1}{\sqrt{2k}}$, $\beta = \frac{1}{\sqrt{2k}}$, $\delta = -\frac{1}{\sqrt{2k}}$. Then

$k(x^2+1) = kx^2 + k$ transforms to

$$\left(\frac{1}{\sqrt{2k}}x - \frac{1}{\sqrt{2k}}\right)^2 \left[k\left(\frac{\frac{1}{\sqrt{2k}}x + \frac{1}{\sqrt{2k}}}{\frac{1}{\sqrt{2k}}x - \frac{1}{\sqrt{2k}}}\right)^2 + k\right]$$
$$= k\left(\frac{1}{\sqrt{2k}}x + \frac{1}{\sqrt{2k}}\right)^2 + k\left(\frac{1}{\sqrt{2k}}x - \frac{1}{\sqrt{2k}}\right)^2$$
$$= \left(\frac{1}{2}x^2 + x + \frac{1}{2}\right) + \left(\frac{1}{2}x^2 - x + \frac{1}{2}\right)$$
$$= x^2 + 1$$

If $Q(x) = ax^2 + bx + c$ with $\Delta = b^2 - 4ac < 0$ and $Q(x) < 0$ for all x then $q < 0$. By 3(i) $Q(x) \approx \frac{4ac-b^2}{4a}(x^2+1)$ where

$$k \equiv \frac{4ac - b^2}{4a} < 0 \qquad \text{(since } a < 0\text{)}.$$

So it is enough to show that $k(x^2+1) \approx -x^2 - 1$ when $k < 0$. Let $\alpha = \gamma = \frac{1}{\sqrt{-2k}}$, $\beta = \frac{1}{\sqrt{-2k}}$, $\delta = -\frac{1}{\sqrt{-2k}}$. As above $k(x^2+1)$ transforms to $-x^2 - 1$. Let $Q(x) = ax^2 + bx + c$ with $\Delta > 0$. By 3(ii) $Q(x) \approx \frac{b^2-4ac}{4a}(x^2-1)$. Let $k = \frac{b^2-4ac}{4a}$. Let $\alpha = \beta = \gamma = -\delta$. Then $k(x^2-1) = kx^2 - k$ transforms to

$$(\alpha x - \alpha)^2 \left(k\left(\frac{\alpha x + \alpha}{\alpha x - \alpha}\right)^2 - k\right) =$$
$$k(\alpha x + \alpha)^2 - k(\alpha x - \alpha)^2 =$$
$$k(\alpha^2 x^2 + 2\alpha x + \alpha^2) - k(\alpha^2 x^2 - 2\alpha x + \alpha^2) =$$
$$4k\alpha x.$$

Let $\alpha = \frac{1}{4k}$ to obtain x.

Let $Q(x) = ax^2 + bx + c$ with $\Delta = 0 = b^2 - 4ac$. By 3(iii) $Q(x) \approx x^2$ when $a > 0$, i.e. when $Q(x) \geq 0$ and $Q(x) \approx -x^2$ when $a < 0$, i.e. when $Q(x) \leq 0$. Let $\alpha = 0$, $\beta = 1$, $\gamma = 1$, $\delta = 0$ and transform x^2 to

$$(x+0)^2\left(\frac{0x+1}{1x+0}\right)^2 = x^2\left(\frac{1}{x^2}\right) = 1.$$

Let $\alpha = 0, \beta = 1, \gamma = 1, \delta = 0$ and transform $-x^2$ to -1. When $Q(x) \equiv 0$ then $Q(x) \approx 0$ is trivial.

$$T = \{ax^2 + bxy + cy^2 \mid a, b, c \in R\}.$$

9. $Q(x,y) = ax^2 + bxy + cy^2$

$$P(x,y) \sim Q(x,y) \text{ if } P(x,y) = Q(\alpha x + \beta y, y) \quad \alpha \neq 0.$$

ERROR

ANSWER:

(i) $ax^2 + bxy + cy^2 \sim \frac{4ac-b^2}{4a}(x^2+y^2)$ when $\Delta = b^2 - 4ac < 0$

(ii) $ax^2 + bxy + cy^2 \sim \frac{b^2-4ac}{4a}(x^2-y^2)$ when $\Delta = b^2 - 4ac > 0$ and $a \neq 0$

(iii) $ax^2 + bxy + cy^2 \sim x^2$ when $b^2 - 4ac = 0$ and $a > 0$
$ax^2 + bxy + cy^2 \sim -x^2$ when $b^2 - 4ac = 0$ and $a < 0$

(iv) $ax^2 + bxy + cy^2 \sim xy + cy$ when $a = 0, b \neq 0$

(v) $ax^2 + bxy + cy^2 \sim cy^2$ when $a = b = 0$.

Transform $Q(x, y) = ax^2 + bxy + cy^2$ to

$$\begin{aligned} Q(\alpha x + \beta y, y) &= a(\alpha x + \beta y)^2 + b(\alpha x + \beta y)y + cy^2 \\ &= a\alpha^2 x^2 + (2a\alpha\beta + \alpha b)xy + (a\beta^2 + b\beta + c)y^2. \end{aligned}$$

(i) Assume $\Delta = b^2 - 4ac < 0$. Then $a \neq 0$. Let $\beta = -\frac{b}{2a}$ then $2a\alpha\beta + \alpha b = 0$ and

$$a\beta^2 + b\beta + c = \frac{-b^2 + 4ac}{4a}.$$

Let $a\alpha^2 = \frac{-b^2+4ac}{4a}, \alpha^2 = \frac{-b^2+4ac}{4a^2}$. This gives

$$Q(\alpha x + \beta y, y) = \frac{4ac - b^2}{4a}(x^2 + y^2)$$

(ii) Assume $\Delta = b^2 - 4ac > 0, a \neq 0$. Let $\beta = -\frac{b}{2a}$ then $2a\alpha\beta + \alpha b = 0$, and $a\beta^2 + b\beta + c = \frac{-b^2+4ac}{4a}$.
Let $a\alpha^2 = \frac{b^2-4ac}{4a}$ to obtain

$$Q(\alpha x + \beta y, y) = \frac{b^2 - 4ac}{4a}(x^2 - y^2)$$

(iii) is as above.
(iv) $a = 0, b \neq 0$. Then

$$Q(\alpha x + \beta y, y) = \alpha bxy + (b\beta + c)y^2.$$

Let $\alpha = \frac{1}{b}$ and $\beta = 0$ to obtain

$$Q(\alpha x + \beta, y) = xy + cy^2.$$