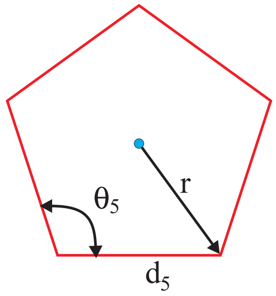
Solutions to 2003 Power Team Exam

Before discussing the solutions to the problems we introduce some notation. Let n denote the number of sides of the regular polygon. (In the picture below $n = 5$.) Let d_n denote the length of a side of a regular n-gon with radius r. Let θ_n denote the interior angle determined by two adjacent sides of this n-gon. The following formulas can be easily derived

$$\theta_n = \frac{n-2}{n}\pi, \quad d_n = 2r\cos\left(\frac{\theta_n}{2}\right)$$



1. Give a formula of the form $y = f(x)$ for the centroidal curve in the case when $n = 5$ and $r = 1$, where r is the distance from a vertex to the centroid. Assume at the start that the wheel has a flat side on the ground, which is the x-axis, and the left vertex of this side is at the origin. You only need supply a formula for x between 0 and 4. Hint: $f(0) = 1$ and for $0 \le x \le \cos(3\pi/10)$ we have $f(x) = \sqrt{1-x^2}$.

   Since the only time the polygon will be moving is when it is rotating about one of its vertices, the path of the centroid must consist of circular arcs with radius equal to the radius of the n-gon. In this case $r = 1$. We impose a coordinate system such that at time $t = 0$ one of the vertices of the polygon is at the origin and the center of the polygon is at the point $(0, 1)$. Then we have

$$\theta_5 = \frac{3\pi}{5} \qquad d_5 = 2\cos(3\pi/10) \approx 1.176$$

   Suppose that we let the polygon roll until one of its sides hits the x-axis. Then the center will be at the point

$$\left(d_5/2, \sqrt{1-(d_5/2)^2}\right) = \left(\cos(3\pi/10), \sqrt{1-\cos^2(3\pi/10)}\right) \approx (0.59, 0.81)$$

   and the path traced out by the centroid is described by

$$f(x) = \sqrt{1-x^2} \text{ for } 0 \le x \le d_5/2\,.$$

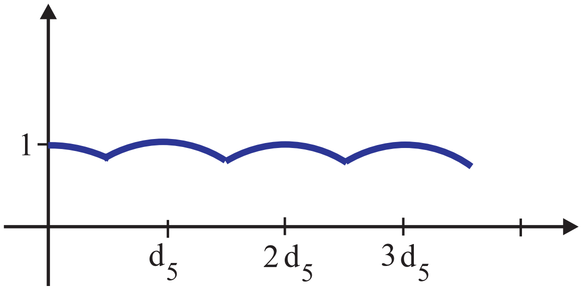
   The polygon then rotates about the vertex located at the point $(d_5, 0)$ until the next side hits the x-axis, the centroid will trace out a circular arc centered at this point of radius 1. A parametric description of the centroid after this rotation is

$$f(x) = \begin{cases} \sqrt{1-x^2}, & 0 \le x \le d_5/2 \\ \sqrt{1-(x-d_5)^2}, & d_5/2 < x < 3d_5/2\,. \end{cases}$$

In general we have

$$f(x) = \begin{cases} \sqrt{1-x^2}, & 0 \le x \le d_5/2 \\ \vdots & \\ \sqrt{1-(x-kd_5)^2}, & (2k-1)\dfrac{d_5}{2} < x \le (2k+1)\dfrac{d_5}{2} \\ \vdots & \end{cases}$$

A plot of the path of the centroid of a 5-gon with radius equal to 1 is shown below.



$d_5 \approx 1.176$

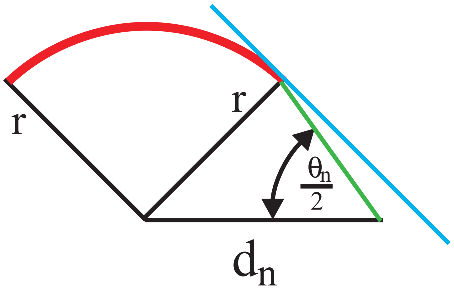
2. Repeat problem 1 for the general case $k > 4$ and $r > 0$. In this problem give the formula for $f(x)$ x between 0 and $4r$.

   The formula for an arbitrary regular n-gon of radius r is similar to that found in question 1.

$$f(x) = \begin{cases} \sqrt{r^2-x^2}, & 0 \le x \le d_n/2 \\ \vdots & \\ \sqrt{r^2-(x-kd_n)^2}, & (2k-1)\dfrac{d_n}{2} < x \le (2k+1)\dfrac{d_n}{2} \\ \vdots & \end{cases}$$

In questions 3-6, assume the wheel is a perfectly balanced regular polygon with n sides $n \ge 3$ and radius $r > 0$, and that the wheel is rolling on a perfectly level surface, subject to no forces other than gravity, and the earth's reactive force when a flat side of the wheel hits the ground. Assume that the wheel does not bounce, slide or skip as it rolls on the flat surface. Assume also that as the wheel rolls, the center of the wheel moves along the centroidal curve in the same way a particle slides friction free along the centroidal curve.

3. To see that the polygon cannot roll if the number of sides is less than 5, we determine the direction of the velocity vector (tangent line) to the centroid's path when the polygon is laying on one of its sides. Since the polygon does not slide, the velocity vector must point past the vertex if the polygon is going to roll over to its next side; for if this vector points towards the vertex or behind it the polygon cannot roll.



In the figure above, the horizontal line represents the side of the polygon, which has just hit the ground, the curved red line is the path of the centroid, the blue line is tangent to this path, and the green line connects the centroid to the vertex, about which the polygon will attempt to pivot. The slope of this latter line is $\tan(\pi - \theta_n/2) = -\tan(\theta_n/2)$, and the slope of the tangent line is $-1/\tan(\theta_n/2)$. Remember that the tangent line is perpendicular to the radial line which has slope $\tan(\theta_n/2)$. If the tangent line is to point past the vertex we must have its slope greater than the slope of the line joining the centroid to the vertex. Thus,

$$\begin{aligned} -1/\tan(\theta_n/2) &> -\tan(\theta_n/2) \\ 1 &< \tan^2\left(\frac{\theta_n}{2}\right) \\ 1 &< \tan\left(\frac{\theta_n}{2}\right). \end{aligned}$$

This means we must have $\frac{\pi}{4} < \frac{\theta_n}{2} = \frac{n-2}{2n}\pi$ or

$$\begin{aligned} n &< 2n - 4 \\ 4 &< n\,. \end{aligned}$$

Note: If the two slopes are equal, the tangent line passes through the vertex, and $n = 4$. Our polygon is a square.

4. To determine how much speed the centroid loses when it hits the ground, we resolve the velocity vector at that moment into two components. One is parallel to the radial direction. That is, it points from the centroid to the vertex that just hit the ground. The second component is perpendicular to this direction and is what causes the polygon to continue rolling. If s_r and s_p denote the magnitudes of these two vectors, then s_r is the amount of speed the centroid loses each time a side hits the ground, and s_p is the speed of the centroid immediately after a side hits the ground. Remember, we're assuming that the polygon does not bounce, slide, or skip. Formulas for these two numbers are

$$\begin{aligned} s_r &= s\sin(\theta_n) \\ s_p &= -s\cos(\theta_n)\ , \end{aligned}$$

where s is the speed of the centroid just before a side hits the ground. Remember that the angle between the tangent line to the path of the centroid and the edge of the triangle is $\theta_n - \pi/2$.

5. To determine what speed is necessary in order for the polygon to rotate once more we note that if this is going to happen then the speed of the centroid must be greater than zero when it is as high as it can get. Define the following symbols: h_l is the height of the centroid when a side is on the ground, and h_t is the maximum possible height of the centroid. Thus,

$$\begin{aligned} h_l &= r \sin\left(\frac{\theta_n}{2}\right) \\ h_t &= r\,. \end{aligned}$$

Let s_l be the speed of the centroid just after a side hits the ground, and let s_t be the speed of the centroid when the centroid is at top dead center. From the formula $\frac{s^2}{2} + gh = c$, we have

$$\begin{aligned} \frac{s_l^2}{2} + gh_l &= \frac{s_t^2}{2} + gh_t \\ \frac{s_l^2}{2} + gr\sin\left(\frac{\theta_n}{2}\right) &= \frac{s_t^2}{2} + gr\,. \end{aligned}$$

Now set $s_t = 0$ (the polygon's centroid gets to top dead center and stops), and solve the resulting equation.

$$\frac{s_l^2}{2} + gr\sin\left(\frac{\theta_n}{2}\right) = gr$$

$$\begin{aligned} s_l^2 &= 2gr - 2gr\sin\left(\frac{\theta_n}{2}\right) \\ &= 2gr\left(1 - \sin\left(\frac{\theta_n}{2}\right)\right) \end{aligned}$$

Thus, the speed of the centroid, immediately after a side hits the ground, must be at least as great as

$$s_l = \sqrt{2gr\left(1 - \sin\left(\theta_n/2\right)\right)}\,.$$

The minimum speed necessary to roll one more time is s_m, where s_m satisfies the equation

$$-s_m \cos\theta_n = s_l = \sqrt{2gr\left(1 - \sin\left(\theta_n/2\right)\right)}\,.$$

If the centroid does not have enough speed to roll one more time it will fall back and lose a percentage of its speed, cf., problem 4. The polygon will then try to roll backwards, but will be unable to do so as its speed is too small to do so. It will then fall back and try to roll forward. This rolling back and forth about a vertex will keep on indefinitely.

6. Suppose that $n = 1000$, $r = 1$ meter, and the initial speed is 90 km per hour. How far will the polygon roll starting with a flat side on the ground. Let s denote the original speed, k the number of flops, and s_k the speed after k flops. We have

$$s_k = s\left(-\cos\theta_n\right)^k .$$

We want the smallest value of k such that

$$\begin{aligned} s\left(-\cos\theta_n\right)^k &\le \sqrt{2rg\left(1-\sin\left(\theta_n/2\right)\right)} \\ k &\ge \frac{(1/2)\ln\left(2rg\left(1-\sin\left(\theta_n/2\right)\right)\right)-\ln s}{\ln\left(-\cos\left(\theta_n\right)\right)} . \end{aligned}$$

We have $n = 1000$, $r = 1$ meter, $s = 90\,(1000)\,/\,(3600) = 25$ meters/second, $\theta_{1000} = \dfrac{998}{1000}\pi \approx 3.13531$, $g = 9.8$. Using these values we get

$$k = 397,214 .$$

Thus, the distance travelled is k times the length of a side, which is $2r\cos\left(\dfrac{\theta_n}{2}\right)$ and

$$\begin{aligned} \text{distance} &= 397,214 * 2 * \cos\left(\frac{998}{2000}\pi\right) \\ &\approx 2495.765 \text{ meters} . \end{aligned}$$