Math Contest, Fall 2004

CD EXAM SOLUTIONS.

1. Mary, John and Kevin collectively have 50 dollars. The ratio of the amount Mary has and the total amount John and Kevin have is three to two. John has 4 dollars more than Kevin. How much does Kevin have?

   Solution. Let J and K be the amounts of John and Kevin respectively. Since together they have $\frac{2}{5}$ of the total amount of \$50, the amount they have is $J+K = \frac{50}{5} \times 2 = 20$. Next, since John has 4 more than Kevin, $J = K+4$. Thus $J+K = 20$ becomes $(K+4)+K = 20$, or $2K = 20-4 = 16$, $K = 8$.
   Therefore, Kevin's amount is \$8.

2. Find three positive integers $x, y,$ and z such that $28x + 30y + 31z = 365$.

   Solution. Number of months with 28, 30, 31 day: $x = 1, y = 4, z = 7$.

3. If $f(x) = 0$ when x is a rational number and $f(x) = 1$ if x is an irrational number, find $5f(\pi) - 3f(f(\pi))$.

   Solution. $5 - 3 \cdot 0 = 5$.

4. Two walls and the ceiling of the room meet at right angles at point P.
   A fly is in the air, 1 meter from one wall, 8 meters from another wall, and 9 meters from the point P.
   How many meters is the fly from the ceiling?

   Solution. Let F denote the point where the fly is. Then FP is the large diagonal of the parallelepiped with the dimensions $1, 8, \text{and}\, x$, with $FP = 9$. By Pythagorean Theorem applied twice, $1^2 + 8^2 + x^2 = 9^2$, so $x^2 = 16$, $x = 4$.

5. Solve the inequality $\dfrac{2x+5}{|x+1|} \geq 1$.

   Solution. $\dfrac{2x+5}{|x+1|} - 1 \geq 0$ or $\dfrac{2x+5-|x+1|}{|x+1|} \geq 0$ becomes $\dfrac{2x+5-(-(x+1))}{-(x+1)} \geq 0$ or $\dfrac{3x+6}{-(x+1)} \geq 0$ on $(-\infty, -1)$ and $\dfrac{x+4}{x+1} \geq 0$ on $(-1, \infty)$.
   Answer. $-2 \leq x < -1$, $x > -1$

6. In a geometric sequence, the sum of the first two terms is 4, and the sum of the first three terms is 13. Find all possible values of the difference between the second term and the first term of this sequence.

   Solution. $a + ar = 4, a + ar + ar^2 = 13$, thus $ar^2 = 9$ and also $\dfrac{1+r+r^2}{1+r} = 134$.
   The last one leads to the quadratic equation $4r^2 - 9r - 9 = 0$ which has solutions $r = 3, r = -34$. If $r = 3$, then $a + 3a = 4, a = 1$. In this case $a_1 = 1, a_2 = 3, a_2 - a_1 = 2$. If $r = -\dfrac{3}{4}$ then $a - \dfrac{3}{4}a = 4, a = 16$. In this case $a_1 = 16, a_2 = 16(-3/4) = -12, a_2 - a_1 = -28$.
   Answer. 2 or -28.

7. The altitudes of a triangle are 12, 15, and 20. What is the largest angle of this triangle?

Solution. Denote sides by $a, b,$ and c. Then $12a = 15b = 20c$ or $\frac{a}{5} = \frac{b}{4} = \frac{c}{3}$, thus $a = 5k, b = 4k,$ and $c = 3k$, so the triangle is a right triangle, the largest angle is $90°$ angle.

8. What is the probability that the the cube of an integer selected at random will end with the digits 11?

Solution.

Let us write a number as $100q+r, 0 \le r \le 99$. Then $n^3 = 1000000q^3+30000q^2r+300qr^2+r^3$. The first 3 terms end with 00, so the last two digits of n^3 are the same as for r^3. If r ends with 1, so does r^3, but if r ends with $2, 3, 4, 5, 6, 7, 8, 9,$ or 0, then r^3 ends with $8, 7, 5, 6, 3, 2, 9$,or 0 respectively. Thus, the only values of the last two digits of numbers r whose cubes could possibly end in 11 are $1, 11, 21, 31, ...91$. The cubes of these two-digit numbers end with $01, 31, 61, 91, 21, 51, 81, 11, 41,$ or 91. Thus n^3 ends with 11 only when n ends with 71. So, the probability is 1/100 (one integer in every hundred ends in 71)

9. Consider the function $f(x)$ such that $f(mn) = f(m + n)$ for all real numbers m and n. If $f(2) = 4$, find $f(64)$.

Solution. $4 = f(2) = f(2 \cdot 1) = f(2 + 1) = f(3 \cdot 1) = f(3 + 1) = ...f(63 + 1) = f(64) = 4$

10. Square ABCD has an area of 192 m^2. It is inscribed in square EFGH which has an area of 224 m^2. Point A lies on the side EF, it is x units from E and y units from F. Find the value of the product xy.

Solution. $x^2 + y^2 = AB^2 = 192$. Next, $EF^2 = (x + y)^2 = x^2 + 2xy + y^2 = 224$. Subtracting the first equation from the second, we have $2xy = 32$, so $xy = 16$.

11. How many zeros are there at the end of the number $1 \times 2 \times 3 \times ... \times 30$?
Solution. There are 7 factors of 5 in the product and more than 7 factors of 2, producing 7 zeros at the end of the product.
Answer. 7

12. Find the sides of a rhombus if the ratio of its diagonals is one to two and its area is 12 in^2.

Solution. Let the lengths of the diagonals be x and $2x$. Since the diagonals of a rhombus are perpendicular, the area of a rhombus is equal to $12 = x^2$. Thus the lengths of the diagonals are $x = 2\sqrt{3}$ and $2x = 4\sqrt{3}$ (inches). Finally, using Pythagorean theorem we find that the side is equal to $\sqrt{3 + 12} = \sqrt{15}$ (inches).

Answer. $\sqrt{15}$ (inches).

13. The function is given by the table

x	1	2	3	4	5
f(x)	4	1	3	5	2

If $u_0 = 4$, and $u_{n+1} = f(u_n)$ for $n \ge 0$, what is the value of u_{1234}?

Solution. Checking the pattern for the first few values we see that $u_0 = 4, u_1 = f(4) = 5, u_2 = f(5) = 2, u_3 = 1, u_4 = f(1) = 4$, thus $u_{4j+k} = u_k, u_{1234} = u_2 = 2$.

14. The difference $D = \sqrt{|40\sqrt{2} - 57|} - \sqrt{40\sqrt{2} + 57}$ is an integer. Knowing that $40\sqrt{2} - 57 < 0$, find the integer D.

Solution. $x = \sqrt{|40\sqrt{2} - 57|} - \sqrt{40\sqrt{2} + 57} = \sqrt{57 - 40\sqrt{2}} - \sqrt{40\sqrt{2} + 57}$. Thus $x^2 = 114 - 2\sqrt{49} = 100$ and so $x = \pm 10$. Finally, since $\sqrt{40\sqrt{2} + 57} > \sqrt{57 - 40\sqrt{2}}$, we conclude that $x < 0, x = -10$.

15. Consider the system of equations:

$$x + y = a, 2x - y = 3$$

Let (x, y) be a solution of this system. Find all values of a such that $x > y$.

Solution. Solve for x, y in terms of a: $x = \dfrac{a+3}{3}$, $y = \dfrac{2a-3}{3}$, so $\dfrac{a+3}{3} > \dfrac{2a-3}{3}$ gives $a < 6$.

16. Solve the system $A^2 + B^2 = 6C, B^2 + C^2 = 6A, C^2 + A^2 = 6B$. What are all possible values of A?

Solution. It is clear that $A, B, C \geq 0$. Next, subtract: $A^2 - C^2 = 6C - 6A$ or $(A - C)(A + C + 6) = 0$, so $A - C = 0, A = C$. Analogously, $B = C$. So $2A^2 = 6A$, and hence $A = B = C = 0$ or $A = B = C = 3$.

Answer. $A = 0$, $A = 3$.

17. Find all integer solutions of the equation $xy - 10(x + y) = 1$.

Solution. $y = \dfrac{10x+1}{x-10} = \dfrac{10(x-10)+101}{x-10} = 10 + \dfrac{101}{x-10}$. Since 101 is a prime, y is an integer only when $x - 10$ is equal to ± 1 or ± 101.

Answer. The integer solutions are $(-91, 9), (9, -91), (11, 111), (111, 11)$.

18. John was travelling on a tourist bus. At some moment he saw a road sign: Alpha Town-125 km, Beta City -175 km. Later, at 11:30 a.m., he asked the driver, "When will we arrive at Beta City?" The driver said, "Well, it depends on how much time you would spend for lunch at Alpha Town. Anyway, it takes 1 hour to travel to Alpha Town and another hour to Beta City". Assuming that the bus travelled at the same speed, when did John see the road sign?

Solution. Let R denote the road stop, D - the point where driver spoke, A-Alpha town, B-Beta City. Then $DA = AB = 175 - 125 = 50$ km. So the speed of the bus is 50 km/h. From R to D, travel time is then $\dfrac{125-50}{50} = 1.5$ hour. Therefore John saw the sign at $11.5 - 1.5 = 10$ a.m.

19. It took Steve 2.5 hours to travel from town X to town Y. His average speed for the whole journey was 80 km/h. For the first 1/4 of the distance, he travelled at an average speed of 60 km/h. Find his average speed for the second part of the journey.

Solution.

Journey	Distance in km	Time in h	Speed in km/h
X to Y (whole trip)	$2.5 \times 80 = 200$	2.5	80
First 1/4 of the trip	$(200 \times (1/4) = 50km$	50/60=5/6 h	60
Second part of the trip	200-50	2.5-5/6	(150/(1 2/3) km/h=90 km/h

20. In an isosceles triangle ΔABC $(AB = BC)$ the altitudes to the base and to the side are equal respectively 10 and 12 cm. Find the length of the base.

Solution. $AB = BC, BD \perp AC, AE \perp BC$, $BD = 10$ cm, $AE = 12$ cm. Let $AC = x, BC = AB = y$. Since $\Delta AEC \sim \Delta BDC$, we get $\frac{BC}{AC} = \frac{BD}{AE}$ or $\frac{y}{x} = \frac{10}{12} = \frac{5}{6}$. Next, we apply Pythagorean Theorem to $\Delta BDC : BC^2 = BD^2 + DC^2, y^2 = 100 + \frac{x^2}{4}$.
Solve $\frac{y}{x} = \frac{5}{6}$, $y^2 = 100 + \frac{x^2}{4}$ to get $x = 15$.
Answer. $AC = 15$ cm.

21. Two circles I and II are externally tangent. A tangent to the circle I passes through the center of the circle II. The distance from the point of tangency to the center of the circle II is three times the radius of the circle II. What is the ratio of the circumference of circle I to the circumference of circle II?

Solution. Let O_1, O_2 be the centers, and let A be the point of tangency. Then $O_1A = r_1, O_2A = 3r_2, O_1O_2 = r_1 + r_2$. From the right triangle, $2(r_1 + r_2)^2 = {r_2}^2 + 3{r_2}^2$. After simplification, $r_1 = 4r_2$, so $\frac{r_1}{r_2} = 4$.

22. A straight line has equation $4x + y = 8$. Find the coordinates of the points on the line which are equidistant from the x and $y-$axes.

Solution. The distance from the point (x, y) to the $x-$ axes is $|y|$, and to the $y-$axes is $|x|$. Since the point is equidistant from the axes, $x = \pm y$. On the other hand, $4x + y = 8$, so we have two options: $y = x, 5x = 8, x = 8/5$, or $y = -x, 3x = 8, x = 8/3$.
Answer. $(8/5, 8/5)$ and $(8/3, -8/3)$.

23. Let the line RT be tangent to a circle at S. Triangle SAB is inscribed into the circle, and $SA = AB$. If the angle $\angle ASR$ is $80°$, find angle $\angle SAB$.
Solution. Connect the center of the circle, C, with the points S, A, and B. Then $CS = CA = CB = r$, where r is the radius of the circle. Since $\angle RSC$ is $90°$, the angle $\angle ASC$ is $90-80 = 10°$. Thus $\angle SAC = 10°$. Since $\triangle SAC = \triangle CAB$ (SSS test), $\angle CAB = \angle SAC = 10°$, and thus $\angle SAB = 20°$.

Answer. $\angle SAB = 20°$.