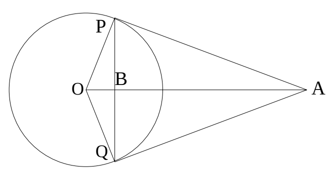
BEST STUDENT EXAM SOLUTIONS
HIGH SCHOOL MATH CONTEST
FALL 2005

1. There are *four* integer solutions. By inspection $x = 3$ is one solution. Also $x = 1$ is a solution since $(1-2)^{25-1^2} = (-1)^{24} = 1$. If $x - 2 \neq 1$ or -1 then $25 - x^2 = 0$ and $x = \pm 5$.

2. ΔOBP is similar to ΔOPA. So

$$\begin{aligned} \frac{\overline{\mathrm{OP}}}{\overline{\mathrm{OA}}} &= \frac{\overline{\mathrm{OB}}}{\overline{\mathrm{OP}}} \\ \frac{r}{\overline{\mathrm{OA}}} &= \frac{\overline{\mathrm{OB}}}{r} \\ \overline{\mathrm{OA}} \cdot \overline{\mathrm{OB}} &= r^2 \end{aligned}$$



3. Now $(a+2b)(a-b) = 10 = 2 \cdot 5$. Since $a + 2b$ is odd, then

$$\begin{aligned} a + 2b &= 5 \\ a - b &= 2\,. \end{aligned}$$

Solving gives $a = 3$ and $b = 1$. So $2a - b = 6 - 1 = 5$.

4.

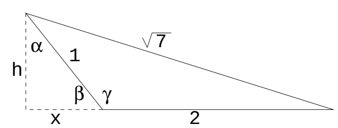
$$\begin{aligned} \log_y x + \log_x y &= 7 \\ \frac{\ln x}{\ln y} + \frac{\ln y}{\ln x} &= 7 \\ \left(\frac{\ln x}{\ln y}\right)^2 + 2\,\frac{\ln x}{\ln y} \cdot \frac{\ln y}{\ln x} + \left(\frac{\ln y}{\ln x}\right)^2 &= 49 \\ (\log_y x)^2 + 2 + (\log_x y)^2 &= 49 \\ (\log_y x)^2 + (\log_x y)^2 &= 47\,. \end{aligned}$$

5.

$$\begin{aligned}
x^2 + h^2 &= 1 \\
(2+x)^2 + h^2 &= 7 \\
(2+x)^2 - x^2 &= 6 \\
4 + 4x &= 6 \\
x &= 1/2
\end{aligned}$$

$$\begin{aligned}
(1/4) + h^2 &= 1 \\
h^2 &= 3/4 \\
h &= \sqrt{3}/2\ .
\end{aligned}$$

So $\alpha = \pi/6$, $\beta = \pi/3$ and $\gamma = \pi - (\pi/3) = (2\pi)/3$.



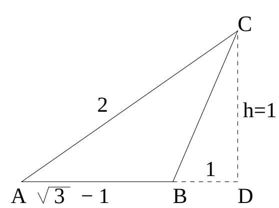
6. The radius of the circle is $300/(2\pi) = 150/\pi$. To travel R feet in one second is to travel $R/(150/\pi) = (\pi R)/150$ radians in one second.

In 50 seconds the R-particle travels $(\pi R/150)50 = (\pi R)/3$ radians and the r-particle travels $(\pi r/150)50 = (\pi r)/3$ radians.

To meet every 50 seconds requires the R-particle to cover 2π radians more than the r-particle, i.e.

$$\begin{aligned}
\frac{\pi R}{3} &= \frac{\pi r}{3} + 2\pi \\
\frac{\pi R}{3} - \frac{\pi r}{3} &= 2\pi \\
R - r &= 6\ .
\end{aligned}$$

7. Since the area of ΔABC is $\sqrt{3} - 1 = (1/2)bh$ then $h = 1$. The right triangle ACD is a $30^\circ, 60^\circ$ right triangle and AD is $\sqrt{3}$ which means BD is 1. So $\angle$CAD is 30°, $\angle$CBD is 45° and $\angle$ABC is $180^\circ - 45^\circ = 135^\circ$. Finally, $\angle$ACB is $180^\circ - (30^\circ + 135^\circ) = 180^\circ - 165^\circ = 15^\circ$.



8. There are *four* subsets of $\{1,2,\ldots,9\}$ that add to greater than 21:

$$\{7,8,9\},\{6,8,9\},\{5,8,9\},\{6,7,9\}.$$

The number of 3×3 arrays having $\{7,8,9\}$ as a row is $3(3!)(6!)$. This is true for $\{6,8,9\}$, etc, so the number of 3×3 arrays having a row that sums to >21 is $(4)(3)(3!)(6!)$. The total number of arrays is $9!$. Hence, the probability is

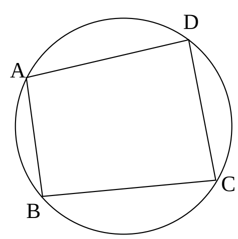
$$\frac{(4)(3)(3!)(6!)}{9!} = \frac{4\cdot 6\cdot 3}{9\cdot 8\cdot 7} = \frac{6\cdot 3}{9\cdot 2\cdot 7} = \frac{1}{7},$$

INPORTANT NOTE: The count above is correct since no 3×3 array can have more than one row adding to a number greater than 21.

9. The opposite angles in an inscribed quadrilateral are supplementary. So $\mathrm{C} = \pi - \mathrm{A}$ and $\mathrm{D} = \pi - \mathrm{B}$. Hence,

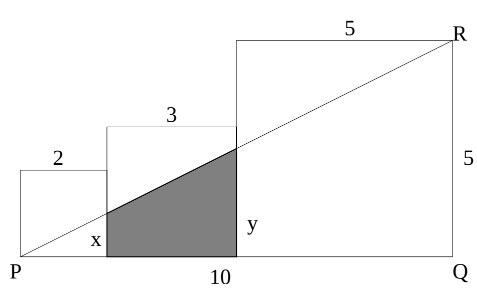
$$\begin{aligned}\sin \mathrm{C} &= \sin(\pi - \mathrm{A}) = \sin\pi\cos\mathrm{A} - \cos\pi\sin\mathrm{A} = \sin\mathrm{A}\\ \cos\mathrm{D} &= \cos(\pi - \mathrm{B}) = \cos\pi\cos\mathrm{B} + \sin\pi\sin\mathrm{B} = -\cos\mathrm{B}\end{aligned}$$

and $\cos\mathrm{B} + \cos\mathrm{D} = 0$. The answer is *two*.



10. Now $\tan(\angle RPQ) = 5/10 = 1/2$. So $1/2 = x/2$ and $x = 1$. Also $1/2 = y/5$ and $y = 5/2$. The area of the trapeziod is

$$\text{Area} = \frac{1}{2}(x+y)3 = \frac{3}{2}\left(1+\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{7}{2} = \frac{21}{4}.$$



11. The answer is *three*. If $x < 4$ then the median is 4 and

$$\frac{1+x+4+6+9}{5} = 4$$

giving one value of x.

If $4 < x < 6$ then the median is x and

$$\frac{1+4+x+6+9}{5} = x\,,$$

giving one more value for x.

If $x > 6$ then the median is 6 and

$$\frac{1+4+6+x+9}{5} = 6$$

leading to the final value of x.

12. Since $45 = 5 \cdot 9$ we seek the smallest positive integer k such that $N - k$ is divisible by both 5 and 9.

For divisibility by 5 the last two digits of $N-k$ could be $40, 35, 30, 25, \ldots, 00$ corresponding to $k = 4, 9, 14, \ldots$

For divisibility by 9 the sum of the digits of $N-k$ must be divisible by 9. We have

$$\begin{array}{rcll} 1+2+\cdots+9 &=& 45 & \text{(divisible by 9)} \\ 1+1+\cdots+1 &=& 10 & \\ 1+2+\cdots+9 &=& 45 & \text{(divisible by 9)} \\ 2+2+\cdots+2 &=& 20 & \\ 1+2+\cdots+9 &=& 45 & \text{(divisible by 9)} \\ 3+3+\cdots+3 &=& 30 & \\ 4+4+4+4 &=& 16 & \\ 1+2+3 &=& 6 & \end{array}$$

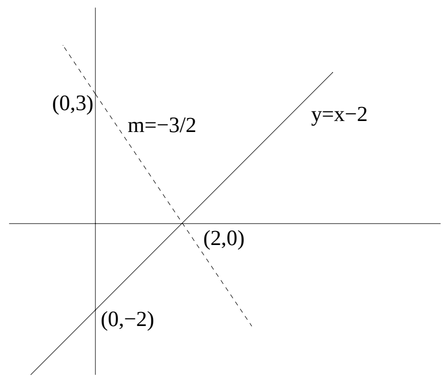
(leaving off the last two digits)

Note that $10+20+30+16+6 = 82$.

Sum of meaningful digits	Last 2	Sum
82	40	82+4+0=86
82	35	82+3+5=90, divisible by

So $N - k = 123\cdots424335$ and $k = 9$.

13. Regardless of m the line $y = mx + 3$ contains the point $(0, 3)$. The line $y = mx + 3$ contains the point $(2, 0)$ when $m = -3/2$. A larger slope, $m > -3/2$, will make the line $y = mx + 3$ intersect $y = x - 2$ at (x_0, y_0) with $x_0 > 0$, $y_0 > 0$, but when $m = 1$, $y = mx + 3$ becomes parallel to $y = x - 2$. For $m > 1$ the line $y = mx + 3$ intersects $y = x - 2$ with a point not in the first quadrant.



14.

$$\begin{aligned} f(x) + 2f(-x) &= \sin x \\ f'(x) - 2f'(-x) &= \cos x \end{aligned}$$

$$\begin{aligned} f'(\pi/4) - 2f'(-\pi/4) &= \cos(\pi/4) = \frac{\sqrt{2}}{2} \\ f'(-\pi/4) - 2f'(\pi/4) &= \cos(-\pi/4) = \frac{\sqrt{2}}{2} \end{aligned}$$

$$\begin{aligned} f'(\pi/4) - 2f'(-\pi/4) &= \frac{\sqrt{2}}{2} \\ -4f'(\pi/4) + 2f'(-\pi/4) &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} -3f'(\pi/4) &= \frac{3\sqrt{2}}{2} \\ f'(\pi/4) &= -\frac{\sqrt{2}}{2}\,. \end{aligned}$$

15. There are *four*.

If $x = 0$ then $|y| = 1$ and $y = \pm 1$. So $(0, 1)$ and $(0, -1)$ are solutions.

Assume $x \neq 0$. Then $1/y = y$ and $y = \pm 1$.

$|x - y| = 1$ means $x - y = 1$ or $x - y = -1$.

If $x-y=1$ then $x=1+y$ and $x=2$. So $(2,1)$ is a solution.
If $x-y=-1$ then $x=y-1$ and $x=-2$. So $(-2,-1)$ is a solution.

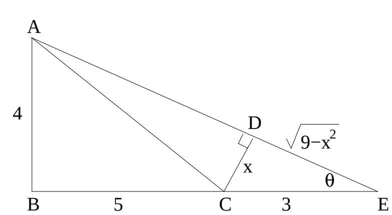
16. The answer is *two*.
At $x=1$, $y=a+b+c<0$
At $x=0$, $y=c>0$.
At $x=0$, $y'=2a(0)+b=b<0$.
$y''=2a>0$, so $a>0$.
Hence, $ab<0$, $ac>0$, $b<0$, $a+b+c<0$ and $a-b+c>0$.

17.

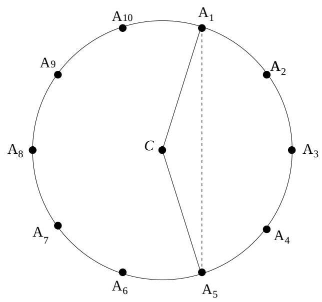
$$\begin{aligned}\tan\theta &= \frac{4}{5+3}=\frac{1}{2}\\ \sin\theta &= \frac{x}{3}\\ \cos\theta &= \frac{\sqrt{9-x^2}}{3}=\frac{1}{2}\end{aligned}$$

So

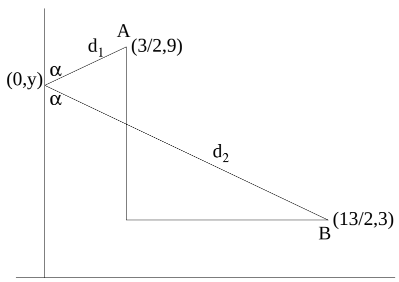
$$\begin{aligned}\tan\theta &= \frac{x}{\sqrt{9-x^2}}=\frac{1}{2}\\ 2x &= \sqrt{9-x^2}\\ 4x^2 &= 9-x^2\\ 5x^2 &= 9\\ x &= \frac{3}{\sqrt{5}}=\frac{3\sqrt{5}}{5}\,.\end{aligned}$$



18. $\angle A_1CA_2 = 360/10 = 36°$. So $\angle A_1CA_5 = (4)(36) = 144°$. Thus $\angle A_1A_5C = (1/2)(180 - 144) = (1/2)(36) = 18°$.



19. The answer is $63/8$.



Let $P = (0, y)$. Then $PA + PB = d_1 + d_2$ is a minimum when the angle of incidence equals the angle of reflection α . So

$$\begin{aligned} \frac{3/2}{d_1} &= \sin\alpha = \frac{13/2}{d_2} \\ 3d_2 &= 13d_1 \end{aligned}$$

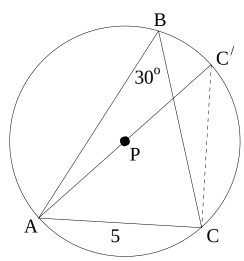
Since $d_1 = \sqrt{(3/2)^2 + (9-y)^2}$ and $d_2 = \sqrt{(13/2)^2 + (3-y)^2}$, then

$$\begin{aligned} 13\sqrt{(3/2)^2 + (9-y)^2} &= 3\sqrt{(13/2)^2 + (3-y)^2} \\ 169((9/4) + (9-y)^2) &= 9((169/4) + (3-y)^2) \\ 169(9-y)^2 &= 9(3-y)^2 \\ \left(\frac{9-y}{3-y}\right)^2 &= \frac{9}{169} \\ \frac{9-y}{3-y} &= \pm\frac{3}{13} \end{aligned}$$

Using $(9-y)/(3-y) = 3/13$ gives $y = 108/10 = 54/5$ which is the point where AB hits the y-axis, not the desired point. Using $(9-y)/(3-y) = -3/13$ gives

$$\begin{aligned}13(9-y) &= -3(3-y) \\ 117 - 13y &= -9 + 3y \\ 126 &= 16y \\ y &= \frac{63}{8} .\end{aligned}$$

20. Let P be the center of the circle



The line segment AC' is a diameter of the circle and $\angle AC'C = 30^\circ$ since it cuts off the same arc of S as does $\angle ABC$. Note that $\Delta AC'C$ is a right triangle with right angle ACC' since AC' is a diagonal. We have

$$\begin{aligned}\sin 30^\circ &= \frac{5}{|AC'|} \\ |AC'| &= \frac{5}{\sin 30^\circ} = \frac{5}{1/2} = 10 .\end{aligned}$$