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2005 Texas A&M Power Team

School: _____

Team Number: _____, Fill in only if your school is submitting more than one entry.

Print the names of all team members in the space below. An ineligible name is cause for disqualification of the team's entry.

Team Members

2005 Texas A&M High School Math Contest
Power Team Questions

Note: clarity of presentation and elegance of solutions will be taken into account when evaluating exams.

Before getting to the problems several terms are defined. Let $f : [a, b] \rightarrow [a, b]$ denote a function defined on the closed interval $[a, b]$, which maps this interval into itself. That is,

$$\text{if } a \leq x \leq b \text{ then } a \leq f(x) \leq b .$$

The terms *fixed point* of, n^{th} *iterate* of, *zero* of, and *orbit* of f are defined below.

A point $\xi \in [a, b]$ is called a fixed point of f , if $f(\xi) = \xi$.

The n^{th} iterate of f , denoted by $f^{(n)}$ is the composition of f with itself n times. That is,

$$\begin{aligned} f^{(0)}(x) &= x \\ f^{(1)}(x) &= f(x) \\ f^{(2)}(x) &= f(f^{(1)}(x)) \\ &\vdots \\ f^{(n+1)}(x) &= f(f^{(n)}(x)) . \end{aligned}$$

An orbit of f is a set of k distinct points $\{x_1, \dots, x_k\}$ in $[a, b]$ such that $x_{i+1} = f(x_i)$ for $i = 1, \dots, k-1$, and $f(x_k) = x_1$. The orbit is said to have length k .

Note that if $f : [a, b] \rightarrow [a, b]$, then the n^{th} iterate of f is defined for all natural numbers n . It is also clear that if $\{x_1, \dots, x_k\}$ is an orbit of f , then each x_i is a fixed point of the k^{th} iterate, $f^{(k)}$, of f .

Define the function f by

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} < x \leq 1 \end{cases} .$$

This function is sometimes referred to as the baker's function as it emulates kneading bread dough. To understand this, picture a roll of bread dough, stretch it to twice its length and then fold it on top of itself so that the two ends overlap. One method of kneading dough is to iterate this process.

1. Show that the function f maps the interval $[0, 1]$ into itself. That is, if $0 \leq x \leq 1$, then $0 \leq f(x) \leq 1$. Thus, the iterates of this function are defined.
2. Find the fixed points of f .
3. Find a formula for $f^{(2)}$ and graph the function.
4. Find the fixed points of $f^{(2)}$.
5. For an arbitrary positive integer n , graph the function $f^{(n)}$. Be sure to clearly show for which values of x we have $f^{(n)}(x) = 1$ or $f^{(n)}(x) = 0$.
6. Find a formula for $f^{(n)}$.

7. For $n = 1, 2, \dots$, let F_n denote the fixed points of $f^{(n)}$. That is,

$$F_n = \left\{ \xi \in [0, 1] : f^{(n)}(\xi) = \xi \right\} .$$

How many points are in F_n and what are they ?

8. Show that the length of any orbit of f that is contained in F_n must divide n .
9. If p is a prime number, show that F_p contains only orbits of length 1 and length p . How many of each type are there?
10. Show that if m divides n , then $F_m \subseteq F_n$. That is, any fixed point of $f^{(m)}$ is also a fixed point of $f^{(n)}$.
11. If r divides n must F_n contain an orbit of length r ?
12. Let m and n be positive integers. How many elements are in the set $F_m \cap F_n$?
13. Let $P_0 = \cup_{n=1}^{\infty} F_n$, $P_1 = f^{-1}(P_0)$, and in general $P_n = f^{-1}(P_{n-1})$. Set $\mathbf{P} = \cup_{n=0}^{\infty} P_n$. The set P_0 consists of all possible periodic points. The sets P_n , for $n > 0$, consists of all points that are periodic after n or less iterations of f . Thus, $1/3 \in P_1$, since $f(1/3) = 2/3 \in F_1 \subseteq P_0$.
 - (a) Show that $P_n \subseteq P_{n+1}$ for $n = 0, 1, \dots$.
 - (b) Characterize the points in \mathbf{P} . That is, give a simple condition on x that is both necessary and sufficient for x to belong to \mathbf{P} .
14. One more item. Suppose you decide to knead some bread dough with repetitions of the following procedure: stretch the dough to twice its original length then turn the first fourth over the second fourth, and the last fourth over the third fourth. See the picture below. What function $f : [0, 1] \rightarrow [0, 1]$ models this procedure?

