2005 Texas A&M High School Math Contest
Solutions to Power Team Questions

Define the function f by

$$f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2-2x, & \frac{1}{2} < x \le 1 \end{cases} .$$

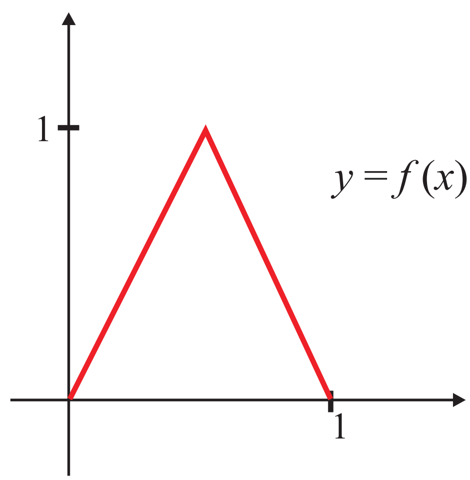
This function is sometimes referred to as the baker's function as it emulates kneading bread dough. To understand this, picture a roll of bread dough, stretch it to twice its length and then fold it on top of itself so that the two ends overlap. One method of kneading dough is to iterate this process.

1. Show that the function f maps the interval $[0,\ 1]$ into itself. That is, if $0 \le x \le 1$, then $0 \le f(x) \le 1$. Thus, the iterates of this function are defined.

   First suppose that $0 \le x \le 1/2$, then $f(x) = 2x$ lies between 0 and 1. If $1/2 < x \le 1$, then

$$\begin{aligned} -2 &\le -2x \le -1 \\ 2-2 &\le 2-2x \le 2-1 \\ 0 &\le f(x) \le 1. \end{aligned}$$

   Thus, f maps $[0,\ 1]$ into $[0, 1]$. The graph of f is shown below.



2. Find the fixed points of f.

   There are two cases to consider. If $0 \le x \le 1/2$, then

$$x = f(x) = 2x \text{ implies } x = 0$$

   if $1/2 < x \le 1$, then

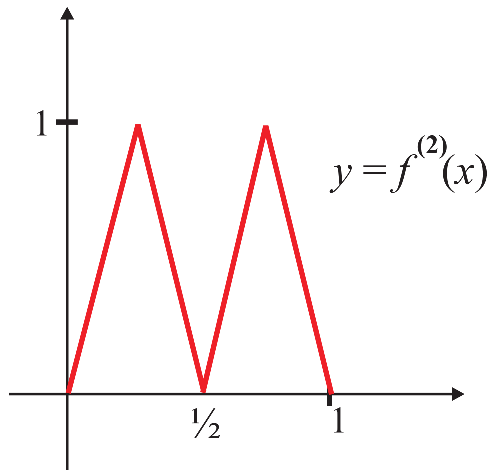
$$x = f(x) = 2 - 2x \text{ implies } x = \frac{2}{3} .$$

   Thus, the fixed points of f are $\{0,\ 2/3\}$.

3. Find a formula for $f^{(2)}$ and graph the function.

$$f^{(2)}(x) = \begin{cases} 4x, & 0 \le x \le 1/4 \\ 2-4x, & 1/4 < x \le 1/2 \\ 4x-2, & 1/2 < x \le 3/4 \\ 4-4x, & 3/4 < x \le 1 \end{cases}$$

The graph of $f^{(2)}$ is shown below.



4. Find the fixed points of $f^{(2)}$.

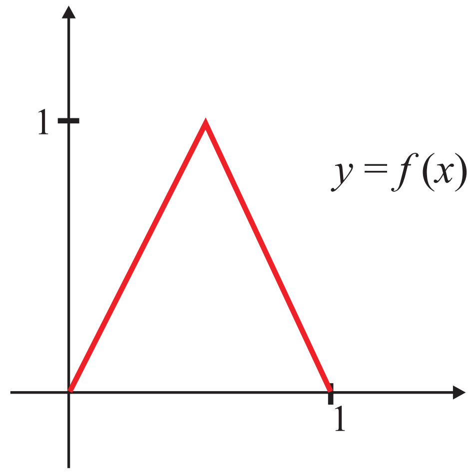
First it is clear that the graph of $y = x$ crosses the graph of $f^{(2)}$ four times. Once in each of the four subintervals where $f^{(2)}$ is a straight line. The four values of x are the solutions to the equations

$$\begin{aligned} x &= 4x, \text{ which implies } x = 0 \\ x &= 2-4x, \text{ which implies } x = \frac{2}{5} \\ x &= 4x-2, \text{ which implies } x = \frac{2}{3} \\ x &= 4-4x, \text{ which implies } x = \frac{4}{5}\ . \end{aligned}$$

Thus, the fixed points of $f^{(2)}$ are $\{0,\ 2/5,\ 2/3,\ 4/5\}$. Note that the fixed points of f are also fixed points of $f^{(2)}$.

5. For an arbitrary positive integer n, graph the function $f^{(n)}$. Be sure to clearly show for which values of x we have $f^{(n)}(x) = 1$ or $f^{(n)}(x) = 0$.

One way to imagine what f does is to picture the graph of f. f takes the vertex at 1 and moves it down to the x-axis and puts each of the two points that are half way between the bottom vertices and the middle of the triangle's base on the line $y = 1$. Compare the graph of f with the graph of $f^{(2)}$. $f^{(3)}$ is the result of f acting on the two triangles composing the graph of $f^{(2)}$. This leads to a total of 4 triangles for $f^{(3)}$ and in general a total of 2^{n-1} triangles for $f^{(n)}$. See the graph of $f^{(n)}$ below.



The base length of each of the triangles is $\frac{1}{2^{n-1}}$. Thus, for

$$x = \frac{k}{2^{n-1}},\ k = 0, 1, \cdots, 2^{n-1},$$

we have $f^{(n)}(x) = 0$. And for those x halfway between these points, that is for

$$x = \frac{k-1}{2^n} + \frac{k}{2^n} = \frac{2k-1}{2^n},\ k = 1,\ 2,\ \cdots,\ 2^{n-1}\ ,$$

we have $f(x) = 1$.

6. Find a formula for $f^{(n)}$.

$$f^{(n)}(x) = \begin{cases} 2^n\left(x - \frac{2k}{2^n}\right), & \frac{2k}{2^n} \le x < \frac{2k+1}{2^n} \\ 2 - 2^n\left(x - \frac{2k}{2^n}\right), & \frac{2k+1}{2^n} \le x < \frac{2k+2}{2^n} \end{cases},$$

for $k = 0,\ 1,\ \cdots,\ 2^{n-1} - 1$.

7. For $n = 1, 2, \cdots$, let F_n denote the fixed points of $f^{(n)}$. That is,

$$F_n = \left\{ \xi \in [0, 1] : f^{(n)}(\xi) = \xi \right\} .$$

How many points are in F_n and what are they ?

The graph of $f^{(n)}$ contains 2^{n-1} triangles, and each triangle has 2 sides that intersect the the line $y = x$. Thus, F_n should contain 2^n points. These points can be found by solving the equations $x = f^{(n)}(x)$. That is,

$$\begin{aligned} x &= 2^n \left(x - \frac{2k}{2^n} \right), \ \frac{2k}{2^n} \le x < \frac{2k+1}{2^n} \text{ and} \\ x &= 2 - 2^n \left(x - \frac{2k}{2^n} \right), \ \frac{2k+1}{2^n} \le x < \frac{2k+2}{2^n} . \end{aligned}$$

The solutions are

$$\begin{aligned} x &= \frac{2k}{2^n - 1}, \text{ and} \\ x &= \frac{2(1+k)}{2^n + 1} , \end{aligned}$$

for $k = 0, 1, \cdots, 2^{n-1} - 1$.

8. Show that the length of any orbit of f that is contained in F_n must divide n.

Let $O = \{x_1, x_2, \cdots, x_k\}$ be an orbit of f of length k that is contained in F_n. That is

$$\begin{aligned} f(x_i) &= x_{i+1}, \text{ for } 1 \le i \le k-1, \text{ and} \\ f(x_k) &= x_1 . \end{aligned}$$

And $x_i \neq x_j$ if $i \neq j$. These equations imply that $f^{(k)}(x_i) = x_i$ for each $x_i \in O$. Since we're assuming the orbit is contained in F_n we also have $f^{(n)}(x_i) = x_i$ for each $x_i \in O$ also. It is clear that $k \le n$. Let q and r be integers such that

$$n = qk + r ,$$

where $0 \le r < k$. Then we have

$$\begin{aligned} x_1 &= f^{(n)}(x_1) = f^{(qk+r)}(x_1) \\ &= f^{(r)} \circ f^{(qk)}(x_1) \\ &= f^{(r)}(x_1) = x_r . \end{aligned}$$

However, this is a contradiction if $r > 0$, as $x_1 \neq x_r$. Thus, k must divide n.

9. If p is a prime number, show that F_p contains only orbits of length 1 and length p. How many of each type are there?

Since the only integer divisors of a prime number are 1 and the number itself, problem 8 tells us that any orbit of f contained in F_p must have length p or 1. An orbit of length 1 is a fixed point of f. Since there are only two fixed points of f there are only 2 orbits of length 1. Moreover, F_p contains 2^p elements so that means that it contains $2^p - 2$ elements that are not fixed points of f. So all of these $2^p - 2$ elements satisfy $f^{(p)}(x) = x$, and each x generates an orbit of length p. Thus, there must be

$$\frac{2^p - 2}{p} = 2\frac{2^{p-1} - 1}{p}$$

orbits of length p in F_p.

10. Show that if m divides n, then $F_m \subseteq F_n$. That is, any fixed point of $f^{(m)}$ is also a fixed point of $f^{(n)}$.

Suppose m divides n. That is, there is an integer l such that $n = lm$. Let $x \in F_m$, then we have

$$\begin{aligned} f^{(n)}(x) &= f^{(lm)}(x) = f^{(m)} \circ f^{(m)} \circ \cdots \circ f^{(m)}(x), \qquad f^{(m)} \text{ appears } l \text{ times} \\ &= f^{(m)} \circ f^{(m)} \circ \cdots \circ f^{(m)}(x), \qquad f^{(m)} \text{appears } l-1 \text{ times} \\ &= f^{(m)}(x) = x\ . \end{aligned}$$

Thus, $x \in F_n$.

11. If r divides n must F_n contain an orbit of length r?

The answer is yes. It is an easy computation to show that if $x_r = \dfrac{2}{1+2^r}$, then x_r satisfies

$$\begin{aligned} f^{(k)}(x_r) &= 2^k x_r \neq x_r, \text{ for } 1 \leq k \leq r-1 \\ f^{(r)}(x_r) &= x_r. \end{aligned}$$

Thus, for any positive integer r, we have found an x that generates an orbit of length r. So if r divides n, then F_n must contain an orbit of length r.

12. Let m and n be positive integers. How many elements are in the set $F_m \cap F_n$?

We show that $F_m \cap F_n = F_r$ where $r = \gcd(m, n)$. That is, r is the greatest common divisor of m and n. Hence $F_m \cap F_n$ contains 2^r elements. To verify that the two sets are equal we note that r divides both m and n. Hence if $x \in F_r$ then $x \in F_m \cap F_n$. Conversely suppose $x \in F_m \cap F_n$. Then x generates an orbit of f of length k. Moreover k must divide both m and n. Hence k divides r, which implies that $x \in F_r$.

13. Let $P_0 = \cup_{n=1}^{\infty} F_n$, $P_1 = f^{-1}(P_0)$, and in general $P_n = f^{-1}(P_{n-1})$. Set $\mathrm{P} = \cup_{n=0}^{\infty} P_n$. The set P_0 consists of all possible periodic points. The sets P_n, for $n > 0$, consists of all points that are periodic after n or less iterations of f. Thus, $1/3 \in P_1$, since $f(1/3) = 2/3 \in F_1 \subseteq P_0$.

(a) Show that $P_n \subseteq P_{n+1}$ for $n = 0, 1, \cdots$.

This is easy to see with the observation that f maps P_0 into P_0. Now, $x \in P_{n+1}$ if and only if $f^{(n+1)}(x) \in P_0$. So if $x \in P_n$, then $f^{(n)}(x) \in P_0$, but then $f^{(n+1)}(x) = f\left(f^{(n)}(x)\right)$. Since, $f^{(n)}(x) \in P_0$ and f maps P_0 into P_0, we have $f^{(n+1)}(x) \in P_0$. Thus, $x \in P_{n+1}$, and $P_n \subseteq P_{n+1}$.

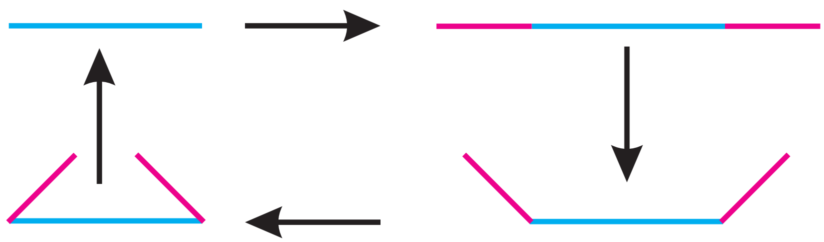
(b) Characterize the points in P . That is, give a simple condition on x that is both necessary and sufficient for x to belong to P .

A point belongs to P if and only if it is rational. It is clear that every point of P_0 is rational since each point in F_n is rational. Moreover, it is clear that f maps rational points into rational points and irrational points into irrational points. Thus, P must be a subset of the rational numbers. To see that it is all of the rationals suppose $x = r/s$ for some integers r and s. Then $f(x)$ is either

$$\frac{2r}{s} \text{ or } 2 - \frac{2r}{s} = \frac{2s - 2r}{s}$$

That is, f maps the rational number onto a rational number with the same denominator. Since there are only a finite number of such rational numbers in the interval $[0,\ 1]$, the numbers x, $f(x)$, $\cdots$, $f^{(n)}(x)$, $\cdots$ must repeat. That is, after a while the values of $f^{(k)}(x)$ must lie in some F_n. This is the statement that x belongs to some P_k and that $x \in$P .

14. One more item. Suppose you decide to knead some bread dough with repetitions of the following procedure: stretch the dough to twice its original length then turn the first fourth over the second forth, and the last forth over the third forth. See the picture below. What function $f : [0,\ 1] \to [0,\ 1]$ models this procedure?



Set f equal to

$$f(x) = \begin{cases} 1/2 - 2x, & 0 \le x \le 1/4 \\ 2x - 1/2, & 1/4 < x \le 3/4 \\ 5/2 - 2x\,, & 3/4 < x \le 1 \end{cases}.$$