2006 Texas A&M High School Math Contest
Power Team Solutions

1. Which is larger $5^{1/3}$ or $\sqrt{2} + \frac{3}{10}$? (Decimal approximations, no matter how many significant figures, will not be considered as justification.)

Solution: Using decimals, we expect $5^{1/3} < \sqrt{2} + \frac{3}{10}$. So assume $5^{1/3} \geq \sqrt{2} + \frac{3}{10}$.
Cube both sides:
$$5 \geq \left(\sqrt{2} + \frac{3}{10}\right)^3 = \sqrt{2}^{\,3} + 3\sqrt{2}^{\,2}\frac{3}{10} + 3\sqrt{2}\,\frac{9}{100} + \frac{27}{1000} = \frac{227}{100}\sqrt{2} + \frac{1827}{1000}$$
Multiply both sides by 1000 and subtract 1827: $5000 - 1827 = 3173 \geq 2270\sqrt{2}$?
Square both sides: $(3173)^2 = 10\,067\,929 \geq (2270)^2 2 = 10\,305\,800$
This is a contradiction.

2. If m and n are positive integers, show that $\sqrt{2}$ lies between $\frac{m}{n}$ and $\frac{m+2n}{m+n}$.

Solution: Since $\frac{m}{n}$ is rational and $\sqrt{2}$ is irrational, either $\frac{m}{n} < \sqrt{2}$ or $\sqrt{2} < \frac{m}{n}$.
Case 1: $\frac{m}{n} < \sqrt{2}$ We need to show $\sqrt{2} < \frac{m+2n}{m+n}$..
$$m^2 < 2n^2 \quad \Rightarrow \quad 2m^2 + 4mn + 2n^2 < m^2 + 4mn + 4n^2 \quad \Rightarrow \quad 2(m+n)^2 < (m+2n)^2$$
$$\Rightarrow \quad \sqrt{2}\,(m+n) < (m+2n) \quad \Rightarrow \quad \sqrt{2} < \frac{m+2n}{m+n}$$
NOTE: This sequence of inequalities was found by working backwards.
Case 2: $\sqrt{2} < \frac{m}{n}$ We need to show $\frac{m+2n}{m+n} < \sqrt{2}$.
$$m^2 > 2n^2 \quad \Rightarrow \quad 2m^2 + 4mn + 2n^2 > m^2 + 4mn + 4n^2 \quad \Rightarrow \quad 2(m+n)^2 > (m+2n)^2$$
$$\Rightarrow \quad \sqrt{2}\,(m+n) > (m+2n) \quad \Rightarrow \quad \sqrt{2} > \frac{m+2n}{m+n}$$

3. Find the largest interval I of positive real numbers such that
$$\left|\frac{1}{x} - \frac{1}{2}\right| < \epsilon,$$
for every $x \in I$, where ϵ is any fixed postive real number.

Solution: The inequality is equivalent to $\frac{1}{2} - \epsilon < \frac{1}{x} < \frac{1}{2} + \epsilon$
We want to take the reciprocal of the three sides of this inequality. We look at three cases:
Case 1: $\epsilon < \frac{1}{2}$
Since, $\frac{1}{2} - \epsilon > 0$, we also have $x > 0$. Therefore, $\dfrac{1}{\frac{1}{2}+\epsilon} < x < \dfrac{1}{\frac{1}{2}-\epsilon}$
So the interval is $I = \left(\dfrac{1}{\frac{1}{2}+\epsilon}, \dfrac{1}{\frac{1}{2}-\epsilon}\right)$
Case 2: $\epsilon > \frac{1}{2}$
If $x > 0$, then $\dfrac{1}{\frac{1}{2}+\epsilon} < x$ with no upper bound.
if $x < 0$, then $x < \dfrac{1}{\frac{1}{2}-\epsilon}$ with no lower bound.

So the interval is either $I = \left(\dfrac{1}{\frac{1}{2}+\epsilon}, \infty \right)$ or $I = \left(-\infty, \dfrac{1}{\frac{1}{2}-\epsilon} \right)$.

Case 3: $\epsilon = \frac{1}{2}$

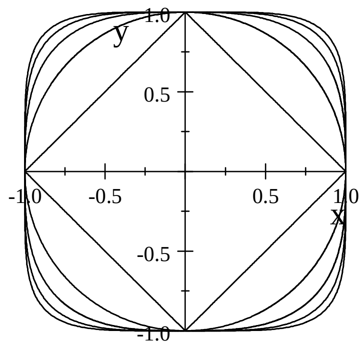
Then $x > 0$ and $\dfrac{1}{\frac{1}{2}+\epsilon} < x$ with no upper bound.

So the interval is $I = \left(\dfrac{1}{\frac{1}{2}+\epsilon}, \infty \right)$

4. For each positive integer n, let $U_n = \{(x,y) : |x|^n + |y|^n \leq 1\}$.

a. Sketch the region U_n.

Solution:



b. How are the U_n related to each other?

Solution: Claim: If $m < n$ then $U_m \subset U_n$.

Proof: Assume $m < n$. If $(x,y) \in U_m$ then $|x|$ and $|y|$ are less than or equal to 1. Hence raising the power we get smaller numbers. Therefore:

$$|x|^n + |y|^n < |x|^m + |y|^m \leq 1.$$

So $(x,y) \in U_n$.

c. Determine the set

$$\bigcup_{n=1}^{\infty} U_n$$

where $\bigcup_{n=1}^{\infty} U_n$ denotes the set consisting of all (x,y) that belong to at least one of the U_n.

Solution: Claim:

$$\bigcup_{n=1}^{\infty} U_n = \{(x,y) \mid -1 < x < 1, -1 < y < 1\} \cup \{(-1,0),(1,0),(0,-1),(0,1)\}$$

where $\{(x,y) \mid -1 < x < 1, -1 < y < 1\}$ is the open square.

Proof: If $(x,y) \in U_n = \{(x,y) : |x|^n + |y|^n \leq 1\}$, then (x,y) is one of the four points $(-1,0),(1,0),(0,-1),(0,1)$ or $|x| < 1$ and $|y| < 1$.

Conversely, the four points $(-1,0),(1,0),(0,-1),(0,1)$ are in U_1 and if $|x| < 1$ and $|y| < 1$ then for some n, $|x|^n < \frac{1}{2}$ and $|y|^n < \frac{1}{2}$. So $|x|^n + |y|^n \leq 1$ and $(x,y) \in U_n$.

5. Let a_k and b_k for $k = 1, 2, \cdots, n$ be arbitrary real numbers. Prove the following formula

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1 \le k < j \le n} (a_k b_j - a_j b_k)^2$$

***Solution**:* We start with the second term on the right and extend the sum to include all j and k. Notice the terms for $j < k$ (which are not included) are equal to those with $k < j$ and the terms with $j = k$ are zero. Thus:

$$\sum_{1 \le k < j \le n} (a_k b_j - a_j b_k)^2 = \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} (a_k b_j - a_j b_k)^2$$

We expand the square and distribute the sums:

$$\sum_{1 \le k < j \le n} (a_k b_j - a_j b_k)^2 = \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} (a_k^2 b_j^2 - 2 a_k b_j a_j b_k + a_j^2 b_k^2)$$

$$= \frac{1}{2} \sum_{k=1}^{n} a_k^2 \sum_{j=1}^{n} b_j^2 - \sum_{k=1}^{n} a_k b_k \sum_{j=1}^{n} a_j b_j + \frac{1}{2} \sum_{j=1}^{n} a_j^2 \sum_{k=1}^{n} b_k^2$$

After changing the j to a k the first and last terms become equal and the two factors in the middle term become equal:

$$\sum_{1 \le k < j \le n} (a_k b_j - a_j b_k)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2$$

6. Show that

$$\sum_{k=1}^{n} |a_k||b_k| \le \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}$$

***Solution**:* We start with the identity from problem 5 and drop the last term which is positive:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

Since this is true for all a_k and b_k, we replace a_k by $|a_k|$ and b_k by $|b_k|$:

$$\left(\sum_{k=1}^{n} |a_k||b_k|\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

Taking the square root of both sides gives our result.

7. Given a collection of n positive numbers a_i, $1 \le i \le n$ there are three standard "averages" associated with them. They are:

$$\text{Harmonic Mean} = \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} = H_n$$

$$\text{Geometric Mean} = (a_1 \cdots a_n)^{1/n} = G_n$$

$$\text{Arithmetic Mean} = \frac{a_1 + \cdots + a_n}{n} = A_n$$

Show, for any finite collection of positive numbers a_i, $1 \le i \le n$ that the following inequalities are true:

$$\min\{a_1, \cdots, a_n\} \le H_n \le G_n \le A_n \le \max\{a_1, \cdots, a_n\}$$

Then show that any of these inequalities can be replaced with equality if and only if $a_1 = a_2 = \cdots = a_n$.

***Solution**:* We break it up into several parts:

Part 1: $G_n \le A_n$

Proof: This is the special case of problem 8 in which every $r_i = \frac{1}{n}$. So we won't prove it separately. Several good direct proofs can be found at http://en.wikipedia.org/wiki/Inequality_of_arithmetic_and_geometric_means

Part 2: $H_n \le G_n$

Proof: Apply the $G_n \le A_n$ inequality to the numbers $\frac{1}{a_1}, \cdots, \frac{1}{a_n}$:

$$\left(\frac{1}{a_1} \cdots \frac{1}{a_n}\right)^{1/n} \le \frac{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}{n}$$

Since both sides are positive, taking reciprocals reverses the inequality and yields the result $H_n \le G_n$.

Part 3: $A_n \le \max\{a_1, \cdots, a_n\}$

Proof: Let $a_{\max} = \max\{a_1, \cdots, a_n\}$. then each $a_k \le a_{\max}$ and

$$a_1 + \cdots + a_n \le a_{\max} + \cdots + a_{\max} = n\,a_{\max}$$

Dividing by n yields the result.

Part 4: $\min\{a_1, \cdots, a_n\} \le H_n$

Proof: Let $a_{\min} = \min\{a_1, \cdots, a_n\}$. then each $a_k \ge a_{\min}$ and each $\frac{1}{a_k} \le \frac{1}{a_{\min}}$. So

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} \le \frac{1}{a_{\min}} + \cdots + \frac{1}{a_{\min}} = \frac{n}{a_{\min}}$$

Dividing by n and taking reciprocals yields the result.

Part 5: $a_1 = \cdots = a_n \quad \Rightarrow \quad \min\{a_1, \cdots, a_n\} = H_n = G_n = A_n = \max\{a_1, \cdots, a_n\}$

Proof: If $a_1 = \cdots = a_n$, then $\min\{a_1, \cdots, a_n\} = \max\{a_1, \cdots, a_n\}$ and the others are squeezed in between.

Part 6: $A_n = \max\{a_1, \cdots, a_n\} \quad \Rightarrow \quad a_1 = \cdots = a_n$

Proof: If $a_1, \cdots, a_n$ are not all equal, then $a_1 + \cdots + a_n < na_{\max}$ and $A_n < \max\{a_1, \cdots, a_n\}$.

Part 7: $G_n = A_n \quad \Rightarrow \quad a_1 = \cdots = a_n$

Proof: This is included with the proof of problem 8. So we won't prove it separately.

Part 8: $H_n = G_n \quad \Rightarrow \quad a_1 = \cdots = a_n$

Proof: Apply part 7 to the numbers $\frac{1}{a_1}, \cdots, \frac{1}{a_n}$.

Part 9: $\min\{a_1, \cdots, a_n\} = H_n \quad \Rightarrow \quad a_1 = \cdots = a_n$

Proof: If $a_1, \cdots, a_n$ are not all equal, then $\frac{1}{a_1} + \cdots + \frac{1}{a_n} < \frac{n}{a_{\min}}$ and $\min\{a_1, \cdots, a_n\} < H_n$.

8. A generalization of the $G_n \le A_n$ inequality is the following: let $r_1, \cdots, r_n$ be postive numbers such that $\sum_{i=1}^n r_i = 1$. If a_i, $1 \le i \le n$ are non-negative numbers, show that

$$\prod_{i=1}^n a_i^{r_i} \le \sum_{i=1}^n r_i a_i$$

where $\prod_{i=1}^n c_i$ denotes the product of the numbers c_1 through c_n, and $\sum_{i=1}^n c_i$ denotes the sum of the numbers c_1 through c_n. Prove this generalization of the $G_n \le A_n$ inequality.

Solution*:* We will prove this by induction. At each stage we will also prove that equality holds only if $a_1, \cdots, a_k$ are all equal. (This is needed for problem 7 Part 7.)

For $n = 1$, we have $r_1 = 1$ and $a_1 = a$. The left side is $\prod_{i=1}^1 a_i^{r_i} = a$ and the right side is $\sum_{i=1}^1 r_i a_i = a$. Equality always holds because there is only one a.

For $n = 2$, we need to show $a^r b^{1-r} \le ra + (1-r)b$ for all $a > 0$ and $b > 0$ with equality only if $a = b$. Divide both sides by b and let $t = \frac{a}{b}$. Then we need to show $t^r \le rt + (1-r)$ for all $t > 0$ with equality only if $t = 1$. Let $f(t) = rt + (1-r) - t^r$. We need to show $f(t) \ge 0$ for all $t > 0$ with equality only if $t = 1$.

Since $f'(t) = r - rt^{r-1}$, we have $f'(1) = 0$ and $t = 1$ is a critical point.

Since $f''(t) = -r(r-1)t^{r-2} = r(1-r)t^{r-2} > 0$ for all $t > 0$, we have f is always concave up and $t = 1$ is an absolute minimum with value $f(1) = 0$. Thus $f(t) \ge 0$ for all $t > 0$ with equality only if $t = 1$.

Now assume the inequality holds for $n = k-1$ and prove it for $n = k$. Let $s = r_1 + \cdots r_{k-1} = 1 - r_k$. Starting with the left hand side, we have:

$$\prod_{i=1}^k a_i^{r_i} = \prod_{i=1}^{k-1} a_i^{r_i} a_k^{r_k} = \left(\prod_{i=1}^{k-1} a_i^{r_i/s}\right)^s a_k^{r_k}$$

Apply the result for $n = 2$ above with $r = s$, $a = \prod_{i=1}^{k-1} a_i^{r_i/s}$ and $b = a_k$:

$$\prod_{i=1}^k a_i^{r_i} \le s \prod_{i=1}^{k-1} a_i^{r_i/s} + r_k a_k \qquad (*)$$

Then using the induction assumption we have:

$$\prod_{i=1}^k a_i^{r_i} \le s\left(\sum_{i=1}^{k-1} \frac{r_i}{s} a_i\right) + r_k a_k = \left(\sum_{i=1}^{k-1} r_i a_i\right) + r_k a_k = \sum_{i=1}^k r_i a_i \qquad (**)$$

To have equality, you need equality in both equations $(*)$ and $(**)$. Thus, for $(**)$ we need $a_1 = \cdots = a_{k-1}$ and for $(*)$ we need

$$a_k = \prod_{i=1}^{k-1} a_i^{r_i/s} = a_1^{(r_1 + \cdots r_{k-1})/s} = a_1$$

9. Suppose Adam buys one dollar's worth of flour each week and Eve buys one pound of flour each week. If the price of flour is not constant from week to week, which one gets the lowest average cost per pound of flour?

***Solution**:* Let $a_1, \cdots, a_n$ be the cost per pound each week.

The total quantity Adam purchases is $\frac{1}{a_1} + \cdots + \frac{1}{a_n}$ which costs him n dollars. So his cost per pound is $\dfrac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}$ which is the harmonic mean H_n.

Eve's total cost is $a_1 + \cdots + a_n$ for n pounds. So her cost per pound is $\frac{a_1 + \cdots + a_n}{n}$ which is the arithmetic mean A_n.

Since $H_n \leq A_n$, Adam has a lower cost per pound than Eve.

10. If $y = f(x)$ is a curve that lies above the x-axis for $a \leq x \leq b$, then the surface area of the surface obtained by rotating the curve about the x-axis is given by the formula

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

Consider the surface S_n obtained by rotating the curve $y = x^n$, $0 \leq x \leq 1$, for $n = 1, 2, \cdots$ about the x-axis. Determine the limit as n tends to infinity of the surface areas of the surfaces S_n.

***Solution**:* If p and q are positive, then

$$\sqrt{p+q} \leq \sqrt{p} + \sqrt{q}$$

Consequently,

$$S_n = 2\pi \int_0^1 x^n \sqrt{1 + (nx^{n-1})^2}\, dx \leq 2\pi \int_0^1 x^n (1 + nx^{n-1})\, dx = 2\pi \left[\frac{x^{n+1}}{n+1} + \frac{x^{2n}}{2} \right]_0^1 = 2\pi \left(\frac{1}{n+1} + \frac{1}{2} \right)$$

On the other hand,

$$S_n = 2\pi \int_0^1 x^n \sqrt{1 + (nx^{n-1})^2}\, dx \geq 2\pi \int_0^1 x^n (nx^{n-1})\, dx = 2\pi \left[\frac{x^{2n}}{2} \right]_0^1 = \pi$$

So for all n,

$$\pi \leq S_n \leq 2\pi \left(\frac{1}{n+1} + \frac{1}{2} \right)$$

By the Squeeze Theorem for limits of sequences, $\lim_{n\to\infty} S_n = \pi$.