High School DE Test: Detailed Solutions

1. Let D be Dick's age and J Jane's age. Then $D = J + 6$ and $D - 6 = 2(J - 6)$. Thus $J = 12$.

2. Let ℓ be the length of the ladder and h the height of its top from the ground. Then by the Pythagorean Theorem, $\ell^2 = h^2 + 15^2$ and $\ell^2 = (h - 13)^2 + 24^2$. Equating the right hand sides of these equations, expanding and simplifying yields $h = 20$. Substituting this into the first equation gives that $\ell = 25$.

3. Denote the numbers by x and y. Then $x + y = 6$ and $\frac{1}{x} + \frac{1}{y} = \frac{2}{5}$. Clear the denominators in the second equation and substitute the first to get

$$xy = \frac{5}{2}(x + y) = \frac{5}{2}6 = 15.$$

   Then

$$\begin{aligned} x^3 + y^3 &= (x + y)(x^2 - xy + y^2) \\ &= (x + y)((x + y)^2 - 3xy) \\ &= 6(36 - 45) \\ &= -54 \end{aligned}$$

4. Let $(-x, 0)$ and $(x, 0)$ be the ends of the base. Then the length of a side of the square is $2x$ and so we must have $2x = \frac{54}{x^2}$. This implies $x = 3$. Hence the area of the square is $(2x)^2 = 36$.

5. Denote by A, B, and C the amounts of money Arnold, Barry and Carl have, respectively. Then $A + B + C = 270$, $\frac{1}{2}B + C = 2A$ and $\frac{1}{3}B + A = C$. Solution of this system yields $A = 75$.

6. Let d be the number of dogs and p the number of people. Then $4d + 2p = 82$ and $d + p = 36$. The solution to this system gives that $d = 5$.

7. Let ℓ, h and w denote the length, height and width, respectively, of the box. Then $\ell w = 120$, $wh = 96$ and $h\ell = 80$. Dividing the first equation by the second gives that $\ell = \frac{120}{96}h = \frac{5}{4}h$. Substitute this into the last equation to get $h\frac{5}{4}h = 80$ and this implies $h = 8$.

8. Denote a typical outcome of this experiment by (a, b), where a and b are in the set $\{1, 2, 3, 4, 5\}$. Since the draws are at random with replacement, each outcome has probability of $\frac{1}{25}$ of occurring. The outcomes giving an even product greater than 10 are $(4, 3), (3, 4), (4, 4), (4, 5), (5, 4)$. Thus the desired probability is $\frac{5}{25} = \frac{1}{5}$.

9. Writing the logs as the difference of the logs of the numerators and denominators, we see the sum is telescoping and so must be equal to $\log 100 - \log 1 = \log 100 = 2$.

10. The expression is equal to $x^2-4x+4+z^2-2z+1+x^2-2xy+y^2+4z^2-4yz+y^2+10$, which in turn is equal to $(x-2)^2+(z-1)^2+(x-y)^2+(2z-y)^2+10$. All the terms except the last are 0 when $x=2$, $y=2$, and $z=1$. Thus the minimum value is 10.

11. Let s be the length of an edge of the cube. Then the octahedron is composed of two pyramids with height $\frac{s}{2}$ and square base with side length of $\frac{s}{\sqrt{2}}$. Hence the volume of the octahedron is $2\cdot\frac{1}{3}\left(\frac{s}{\sqrt{2}}\right)^2\frac{s}{2}=\frac{s^3}{6}$, and so the ratio of the volume of the cube to the volume of the octahedron is 6.

12. We have $(a-1)(b-1)=120$, $(b-1)(c-1)=60$ and $(c-1)(a-1)=72$. Divide the first equation by the second to get $\frac{a-1}{c-1}=2$ and plug this into the third equation to get $(c-1)^2=36$. Thus $c=-5$ or 7. When $c=-5$, we get $a=-11$ and $b=-9$, in which case $a+b+c=-25$. When $c=7$, we have $a=13$, $b=11$ and so $a+b+c=31$.

13. Let x be the length of $\overline{AF}$. Let G be the point of intersection of $\overline{AB}$ with the line through $\overline{EF}$. Then $\overline{AG}=\frac{1}{2}$ and $\overline{FG}=1-x$. By the Pythagorean Theorem, $\left(\frac{1}{2}\right)^2+(1-x)^2=x^2$. Solving for x gives $x=\frac{5}{8}$. Thus the area of ΔAFB is $\frac{1}{2}\cdot 1\cdot(1-x)=\frac{3}{16}$.

14. Let $s=\overline{ZA}=\overline{AC}$ and $t=\overline{ZY}$. Then $2s=\overline{ZC}$ and by the law of cosines, $t^2=s^2+(2s)^2+s(2s)\cos 120^\circ=7s^2$. The triangles in question are equilateral, so

$$\frac{\Delta XYZ}{\Delta ABC}=\frac{t^2}{s^2}=7.$$

15. All the numbers appearing after 1 are larger than 1, so throw out 1. Now $3^2>2^3$, $3^4>4^3$ and $3^5>5^3$ and so we get $3^{1/3}>2^{1/2}$, $3^{1/3}>4^{1/4}$ and $3^{1/3}>5^{1/5}$. Thus $3^{1/3}$ is the largest.

16. This forms a square in the plane with vertices at $(1,0)$, $(-1,0)$, $(0,1)$ and $(0,-1)$. Thus the length of each side is $\sqrt{2}$ and so the area is 2.

17. If x is the score she needs on the final, then we must have

$$\frac{80+81+73+65+91+2x}{7}=80.$$

Solving for x yields $x=85$.

18. Setting $x=2$, 0, 1 will give that $C=0$, $B=1$, and $A=1$.

19. Let $x=\angle CBE$. Then since $\overline{CD}$ is parallel to $\overline{BE}$, $180^\circ-3x=x$. This tells us $x=45^\circ$. Therefore

$$\angle A+(\angle B-45^\circ)+(\angle B-45^\circ)=180^\circ.$$

Solving for $\angle A$ gives $\angle A=270^\circ-2\angle B$.

20. Let x be the thickness of the L. Then the legs of the triangle each have length $1-x$. The area of the square containing the figure is 1 and since the area of the triangle and the L are the same, the area of the triangle must be $\frac{1}{3}$. On the other hand, the area of the triangle is $\frac{1}{2}(1-x)(1-x)$. Equating the two expressions and solving for x yields $x = 1 - \frac{\sqrt{6}}{3}$.

21. Let x be the radius. Denote by O the midpoint of the segment and by M the point of tangency of the circle 2 and the segment. Connect the center of the circle 2 with the center of circle 1 and connect the point N of tangency of the semicircle and circle 2 with the point O. Then x can be found from the equation expressing MO two ways: $MO^2 = (\frac{R}{2}+x)^2 - (\frac{R}{2}-x)^2 = (R-x)^2 - x^2$

    Answer: $\frac{R}{4}$

22. Let X be the center of the smaller circle and let Y be the center of the other one. Let Z be the point along $\overline{YC}$ that is r units from C. Then $|\overline{XY}| = R+r$, and since $\overline{XB} \parallel \overline{YC}$, we must have $\overline{XZ} \parallel \overline{BC}$ and $|\overline{XZ}| = |\overline{BC}|$. Therefore ΔXYZ is a right triangle with sides of length $|\overline{BC}|$, $R-r$ and $R+r$. Thus $|\overline{BC}|^2+(R-r)^2 = (R+r)^2$, and this implies $|\overline{BC}| = 4Rr$.