1999 AB EXAM SOLUTIONS

1. Let A be the area of triangle **ABC**. The area of the shaded region is

$$\frac{1}{4}A+\frac{1}{4}A+\frac{1}{4}A+\frac{1}{16}A=\frac{13}{16}A\ ,$$

so the fraction of the total area that is shaded is $13/16$.

2. Now $2^{100}<5^{50}<3^{75}$, since

$$\begin{aligned} 2^{100} &= 4^{50} < 5^{50} \\ 3^{75} &= 27^{25} \\ 5^{50} &= 25^{25} \qquad \Longrightarrow \qquad 5^{50} < 3^{75} \end{aligned}$$

3. From the table, Boston and Toronto tied. Since Montreal defeated Boston 3-0, this accounts for all 3 goals scored against Boston. So in the Toronto-Boston game, Boston scored 0 goals. Hence there was a 0-0 tie.

4. Let P be the set of applicants taking physics and let C be the set of applicants taking chemistry. Then

$$\begin{aligned} |P\cup C| &= 100-10=90 \\ |P\cup C| &= |P|+|C|-|P\cap C| \\ 90 &= 75+83-|P\cap C| \\ |P\cap C| &= 158-90=68\ . \end{aligned}$$

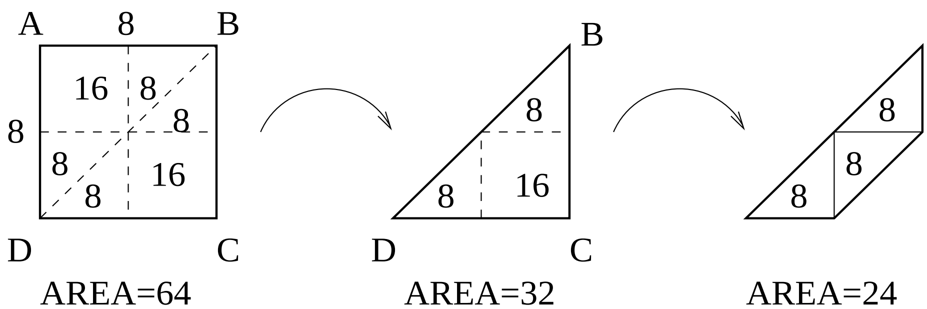
5. Since the sum is even, one of the primes must be 2. So we seek primes p and q with $p+q=38$. By trial and error the primes are 7 and 31. So (7)(31)=217.

6. Let $x=$ # of cards Paul had. Then

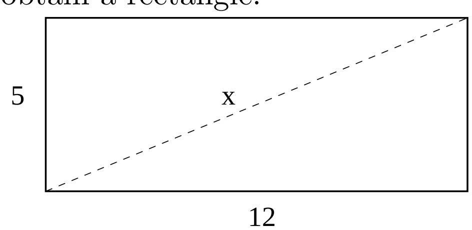
$$\begin{aligned} \frac{1}{2}x+\frac{1}{6}x+12 &= x \\ x-\frac{1}{2}x-\frac{1}{6}x &= 12 \\ \frac{6-3-1}{6}x &= 12 \\ \frac{1}{3}x &= 12 \\ x &= 36\ . \end{aligned}$$

7. $8 \cdot 10^{18} + 1^{18} = 8 \cdot (9+1)^{18} + 1$. Now 9 divides every term in the expansion $(9+1)^{18}$, except the last one, 1, giving a remainder of 1. Since $8 \cdot 1 + 1 = 9$, there is a final remainder of 0.

8.



9. Unfold the cylinder to obtain a rectangle.



$$x^2 = 5^2 + 12^2 = 25 + 144 = 169$$
$$x = 13 \,.$$

10. Now
$$\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \frac{6}{5} \cdots \frac{a}{b} = \frac{a}{2} \,.$$
We need $\dfrac{a}{2} = 9$ or $a = 18$. Then $b = a - 1 = 17$. So $a + b = 35$.

11. $(n-2)(n-1)(n)(n+1)(n+2)$ is the product of 5 consecutive integers. At least two of the 5 must be even and differ by 2. So 8 is a divisor. At least one of the 5 is divisible by 3, so 3 is a divisor. At least one is divisible by 5, so 5 is a divisor. No other prime must divide this product. So (8)(3)(5)=120 is the largest integer that must divide $(n-2)(n-1)(n)(n+1)(n+2)$ for all n.

12.

$$\begin{array}{r} 352 \\ 532 \\ 752 \\ 572 \\ 372 \\ 732 \\ \hline 3312 \end{array}$$

and $3312/6 = 552$.

13. The basketball player must make the first shot and miss the second shot. The probability of such an occurrence is $(3/4)(1/4) = 3/16$.

14. 2Q ends in the digit 4 so $Q \in \{2, 7\}$. If Q=2, then

$$\begin{array}{rr} & 22 \\ \times & \text{S}2 \\ \hline & 44 \\ & xxx \\ \hline & 1404 \end{array}$$

and 2S ends in 6, so $S \in \{3, 8\}$. Neither choice for S gives 1404 as a product. So Q=7 and then

$$\begin{array}{rr} & 27 \\ \times & \text{S}2 \\ \hline & 54 \\ & xxx \\ \hline & 1404 \end{array}$$

and we conclude that S=5. Thus Q+S=12.

15.

$$\begin{array}{l} 3 \\ 3+4 \\ 3+(2)(4) \\ 3+(3)(4) \\ \quad\vdots \\ 3+(21)(4) \\ \hline 3(22)+4(1+2+\cdots+21) \end{array} = 66 + 4\left(\frac{(21)(22)}{2}\right) = 990$$

Hence, $15k = 990$, so $k = 66$.

16. Let r be the radius of the smaller circle. The area of the shaded region is

$$\frac{\pi(2r)^2}{6} - \frac{\pi r^2}{6} = \frac{9\pi}{8} \Longrightarrow \frac{1}{2}\pi r^2 = \frac{9\pi}{8} \Longrightarrow r^2 = \frac{9}{4} \Longrightarrow r = \frac{3}{2}\ .$$

The area of the smaller circle is $\frac{9}{4}\pi$.

17. The area is $(1)(6) + ((1/2)(3)(6)) = 15$.