CD EXAM
WITH PROOFS

1. For what value(s) of the number a do the equations

$$\begin{aligned} x^2 - ax + 1 &= 0 \\ x^2 - x + a &= 0 \end{aligned}$$

have a common real solution?

Proof. Let z be a common real solution. Then

$$\begin{aligned} z^2 - az + 1 &= z^2 - z + a = 0 \\ \Rightarrow (1-a)z &= a - 1 \\ \Rightarrow a &= 1 \text{ or } z = -1 \end{aligned}$$

If $a = 1$, then $x^2 - x + 1 = 0$ has no real solutions. So $z = -1$ and

$$\begin{aligned} (-1)^2 - a(-1) + 1 &= 0 \\ 1 + a + 1 &= 0 \\ a &= -2 \end{aligned}$$

□

2. Find the domain of

$$f(x) = \frac{2}{[x]^2 + [x] - 56}$$

where [ ] is the greatest integer function.

Proof.

$$\begin{aligned} [x]^2 + [x] - 56 \quad &= \quad ([x]+8)([x]-7) \\ [x] + 8 = 0 \quad &\text{when} \quad x \in [-8, -7) \\ [x] - 7 = 0 \quad &\text{when} \quad x \in [7, 8) \end{aligned}$$

The domain of f is $(-\infty, -8) \cup [-7, 7) \cup [8, \infty)$. □

3. Let
$$f(x) = \frac{cx}{2x+3}$$
with $x \neq -3/2$. Find all values of c, if any, for which $f(f(x)) = x$ for all $x \neq -3/2$.

Proof.
$$\begin{aligned} x = f(f(x)) &= f\left(\frac{cx}{2x+3}\right) = \frac{c\left(\frac{cx}{2x+3}\right)}{2\left(\frac{cx}{2x+3}\right)+3} \\ &= \frac{c^2x}{2cx+6x+9} = \frac{c^2x}{(2c+6)x+9} \end{aligned}$$
$$\text{Thus we want } (2c+6)x^2 + 9x = c^2x$$
$$\text{or } (2c+6)x^2 + (9-c^2)x = 0 \text{ for all } x \neq -3/2$$
$$\Rightarrow 2c+6 = 0 \text{ and } 9 - c^2 = 0$$
$$\Rightarrow c = -3$$
□

4. Find the y-coordinate of the point on the y-axis which is equidistant from $(5,-5)$ and $(1,1)$.

Proof. We need the distance $d((0,u),(1,1)) = d((0,u),(5,-5))$.
$$\begin{aligned} 1 + (u-1)^2 &= 25 + (u+5)^2 \\ u^2 - 2u + 1 &= 24 + u^2 + 10u + 25 \\ -12u &= 48 \\ u &= -4 \end{aligned}$$
□

5. A cube of volume 216 cubic inches is inscribed in a sphere. What is the surface area of the sphere?

Proof. Let x be the length (in inches) of a cube edge. The cube has volume $x^3 = 216$, so $x = 6$. If the sphere is centered at $(0,0,0)$, then the inscribed cube has a vertex at $(3,3,3)$. The radius of the sphere is $r = \sqrt{(3)^2 + (3)^2 + (3)^2} = \sqrt{27}$. The surface area of the sphere is $4\pi r^2 = 4\pi(\sqrt{27})^2 = 108\pi \text{in}^2$. □

6. An elastic band is placed around the top of 4 circular cans, each with a radius of 6 inches. Find the length of the stretched band.

Proof. The rubber band can be cut and reconfigured to be a circle of radius 6 inches and a square with side length 12 inches. Thus the length is $[4 \times 12] + [2\pi 6] = 48 + 12\pi$ inches. □

7. Find two distinct (complex) numbers each of which is the square of the other.

Proof. Let z and w be the two numbers. Since $z^2 = w$ and $w^2 = z$, we also know that $z = w^2 = (z^2)^2 = z^4$. Thus, either $z = 0$ or $z^3 = 1$. If $z = 0$, then $z^2 = w$ is also 0 and z and w are not distinct. Thus $z^3 - 1 = (z-1)(z^2+z+1) = 0$. If $z = 1$, then $z^2 = w = 1$ and again z and w are not distinct. z is a solution of $z^2 + z + 1 = 0$, and by the quadratic formula,

$$z = \frac{-1 \pm \sqrt{1-4}}{2}$$

Since the set-up is symmetric with respect to z and w, w is also one of these two numbers. Since z and w are distinct, z is one of them and w is the other. It is now easy to check that each of these numbers is the square of the other one. □

8. A positive integer x is chosen with $10 \leq x \leq 99$. If x is divided by the sum of its digits, how small can the result be?

Proof. Let a and b be the digits of x so that $x = 10a + b$ and $1 \leq a \leq 9$ and $0 \leq b \leq 9$. The problem asks us to maximize and minimize

$$\frac{10a+b}{a+b} = 1 + \frac{9a}{a+b} = 1 + \frac{9}{1+\frac{b}{a}}$$

From the final expression it is clear that to maximize the result the denominator $(1 + \frac{b}{a})$ should be as small as possible. Thus $\frac{b}{a}$ should be as small as possible. Thus b should be as small as possible and a should be as large as possible. By the restrictions on a and b, the max occurs at $x = 90$, and the max is 10.

Similarly, the minimum will make $1 + \frac{b}{a}$ as large as possible. This occurs when b is as large as possible and a is as small as possible. Thus $b = 9$, $a = 1$, $x = 19$, and the min is 1.9. □

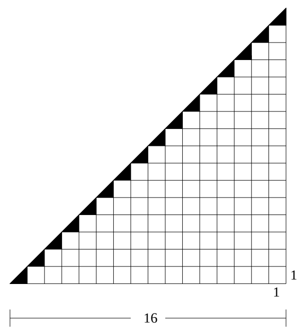
9. The square of the sum of two numbers is 64 and the sum of the squares of the two numbers is 34. Find the product of the two numbers.

Proof. Let x and y be the numbers. We know that $(x + y)^2 = 64$ and that $x^2 + y^2 = 34$. Thus $2xy = 30$ and $xy = 15$. □

10. Find the ratio of the area of the shaded area to the unshaded area.

Proof. Let 1 unit be the length of a square as shown. The shaded area is the same as the total area of 8 such squares. The triangle shown has area

$\frac{1}{2}(16)(16) = 128$ units^2. The ratio of the shaded area to the unshaded area is $\frac{8}{128-8} = \frac{8}{120} = \frac{1}{15}$. □



11. Pick any 5 points in the plane (p, q, r, s, and t) and draw the line segments pq, qr, rs, st, and tp (they are allowed to cross). If a line l is drawn which goes through all 5 of these line segments, how many of the 5 original points must lie on l? In other words, can the 5 points and the line l be drawn so that none of the 5 points lie on l? Can they be drawn so that only 1 of them lies on l, etc.

   Proof. The answer is 1 by a parity argument. If l contains none of the points, then since it crosses pq, then p and q are on opposite sides of l. Since l also crosses qr, rs, and st, then this means that p and t would be on the same side and that l would not cross tp. On the other hand it is easy to draw a picture where l contains only 1 of the five points. □

12. Let A, B, C, D and E denote the vertices of a regular pentagon in the plane. If a line is drawn through each pair of these points, into how many regions is the plane divided?

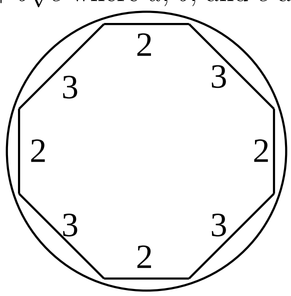
   Proof. There are 36. A careful picture will show them what is going on. They have to be a little careful with various abstract approaches since the arrangement of lines is not generic (i.e. there are parallel lines and there are places where 4 lines intersect at a point) and some of the lines intersect far away from the original pentagon. There are 16 regions inside the pentagon and 25 outside. □

13. Six consecutive integers are written on a blackboard. When one of them is erased, the sum of the remaining five is 1999. What number was erased?

   Proof. If n is the first number, then the others are $n + i$, $i = 1, 2, 3, 4, 5$. The sum of all 6 is $6n + 15$, so the sum of five of them is $5n + 15 - i$ where

$i = 0, 1, 2, 3, 4$ or 5. If $5n + 15 - i = 1999$, then $5n + 15 = 1999 + i$ and $1999 + i$ must be a multiple of 5. The restrictions on i now require that $i = 1$ (i.e. the second number was the one removed) and that $5n = 1985$ or $n = 397$. Thus the number erased was 398. □

14. An octagon is inscribed in a circle. If the lengths of the eight sides are 2, 3, 2, 3, 2, 3, 2, 3, 2 and 3, in that order, find the area of the octagon in the form $a + b\sqrt{c}$ where a, b, and c are integers.



Proof. Notice that if the center of the circle is connected to the vertices of the octagon, then the area is the sum of the 8 triangles formed. Notice also that the area is the same if the order in which the sides occur is rearranged. Thus conside the figure where the sides of length 2 and 3 alternate. If we add a $45 - 45 - 90$ right triangle with hypotenuse 2 to each of the sides of length 2, then the result is a square with side length $3 + 2\sqrt{2}$. Thus the area of the octagon is

$$(3 + 2\sqrt{2})^2 - 4(1/2)(\sqrt{2}^2) = 9 + 12\sqrt{2} + 8 - 4 = 13 + 12\sqrt{2}$$

□