Amoebas and their Tropical Varieties

Timothy Jewell

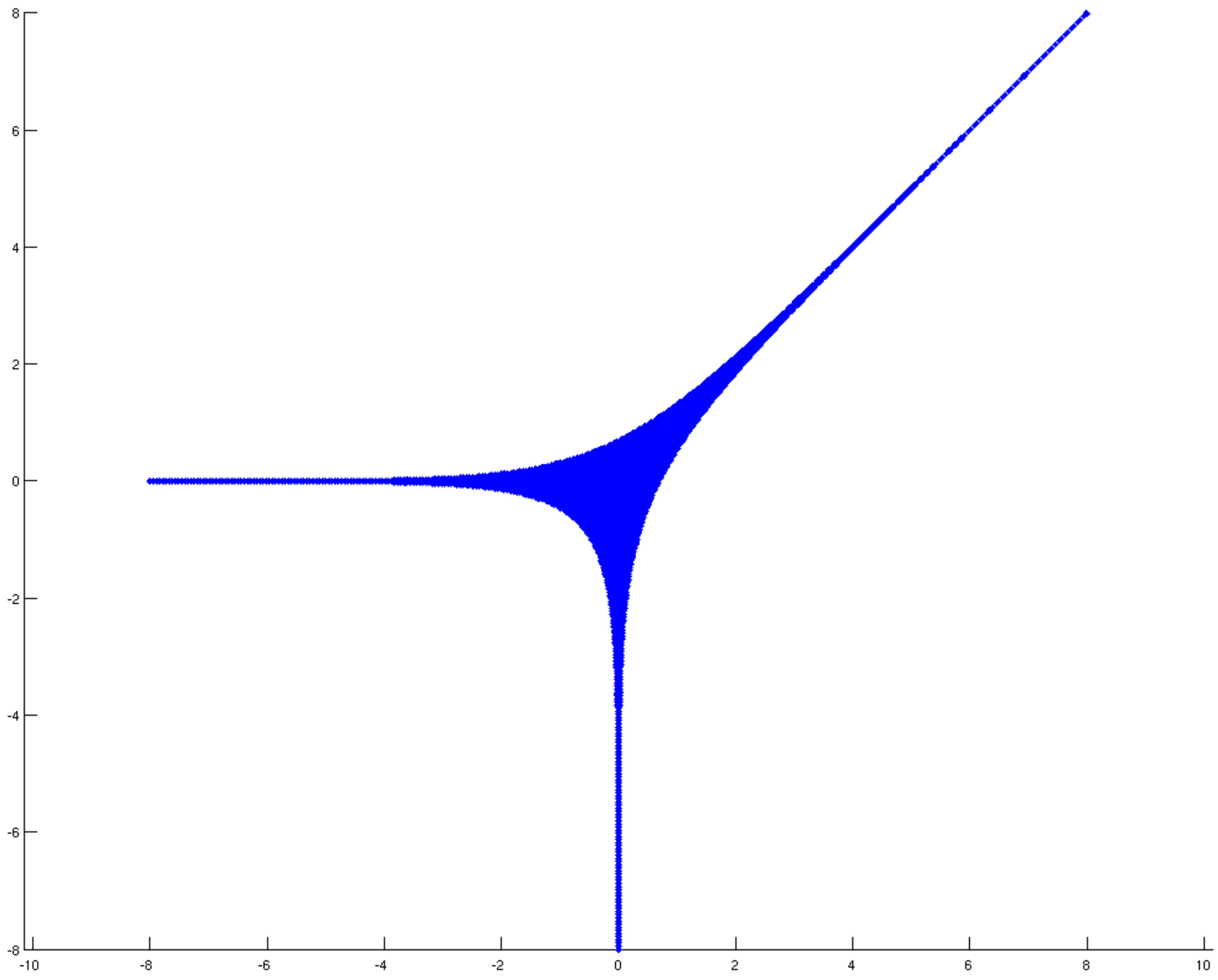
J. Maurice Rojas, Advisor

What is an Amoeba?

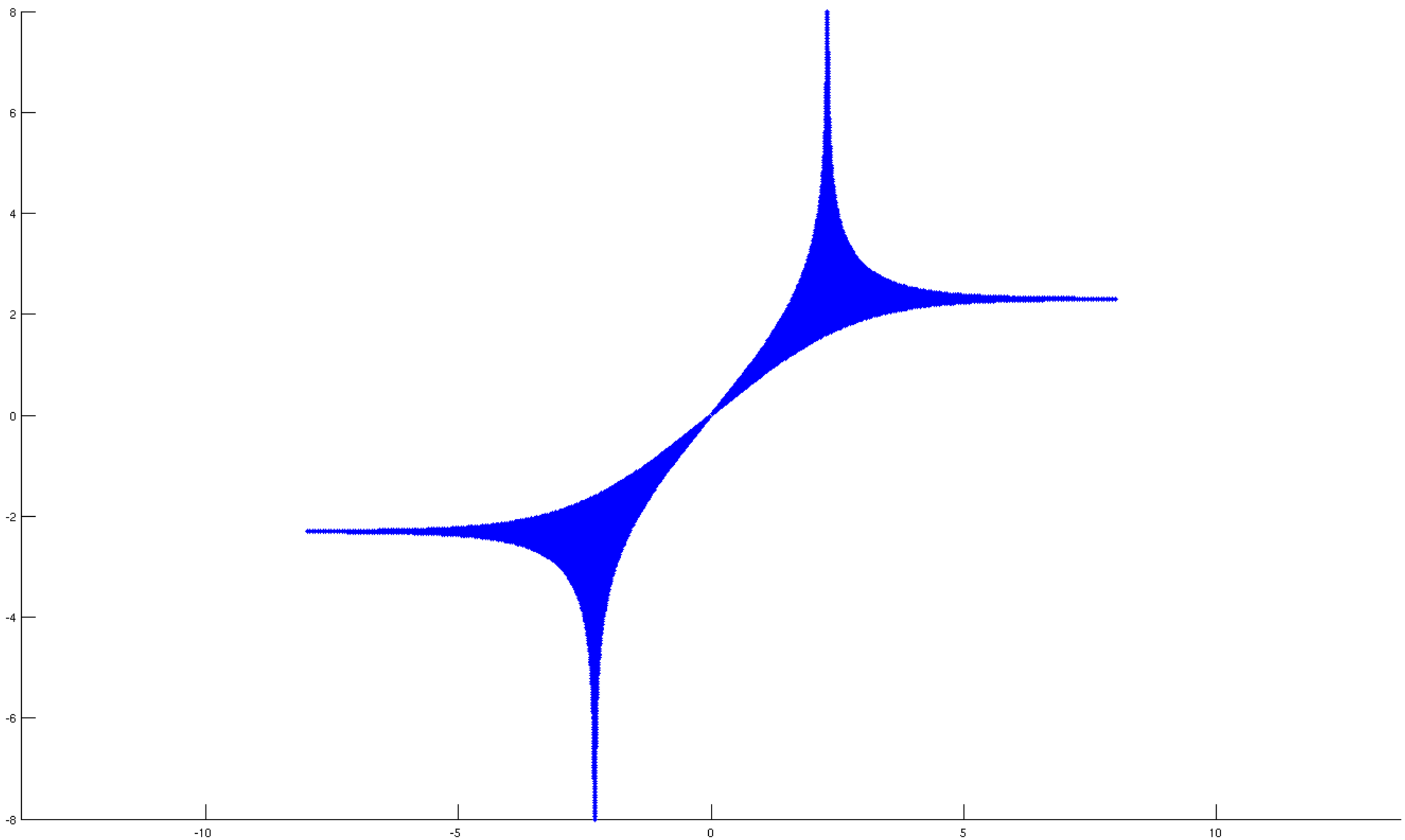
- For any polynomial f in two variables:

$$\text{Amoeba}(f) := \{(\log|x_1|, \log|x_2|) \mid f(x_1, x_2) = 0 \text{ and } x \in (\mathbb{C}^*)^2\}$$

$$f(x_1, x_2) = 1 + x_1 + x_2$$

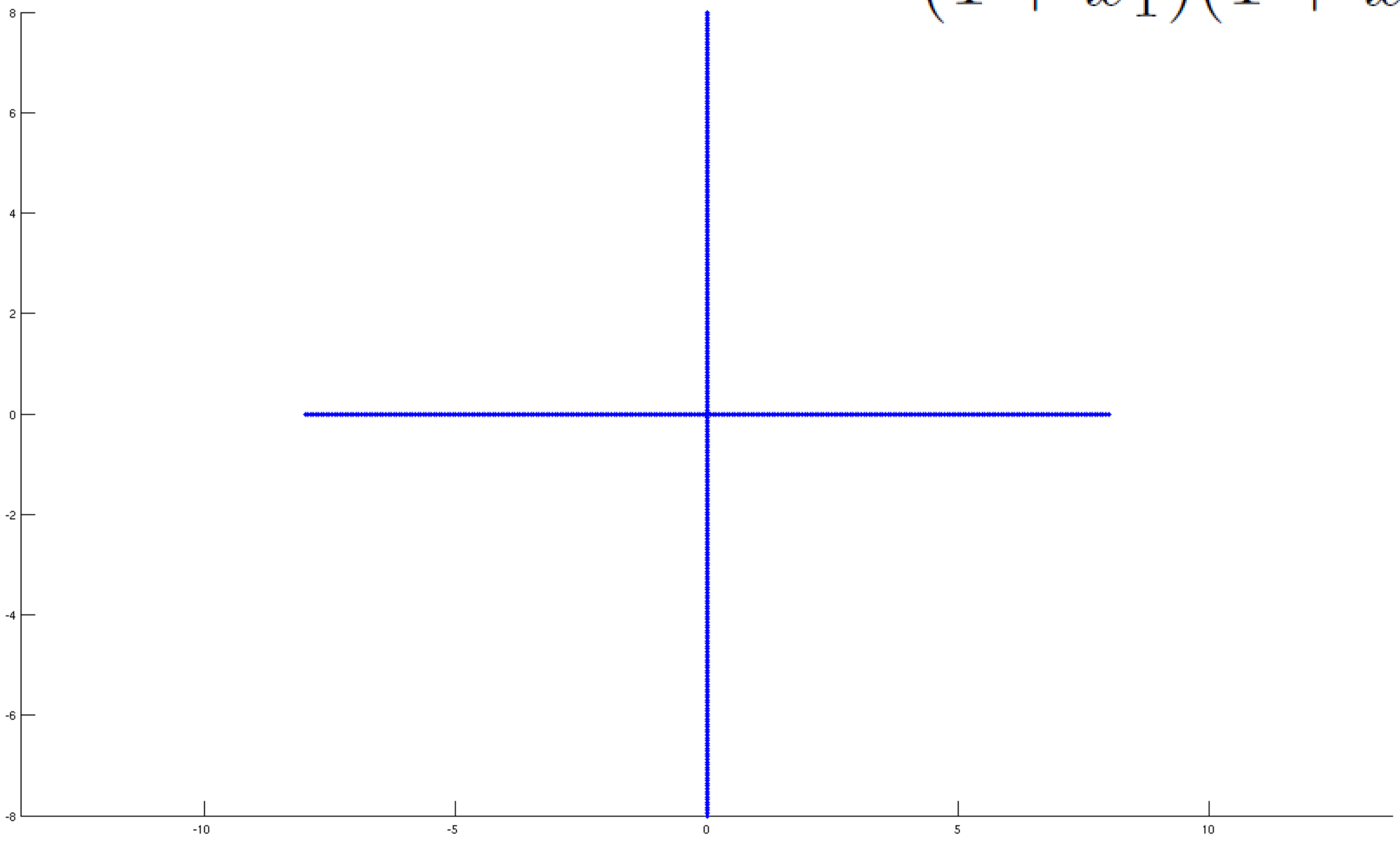


$$f(x_1, x_2) = 1 + 10x_1 + 10x_2 + x_1x_2$$

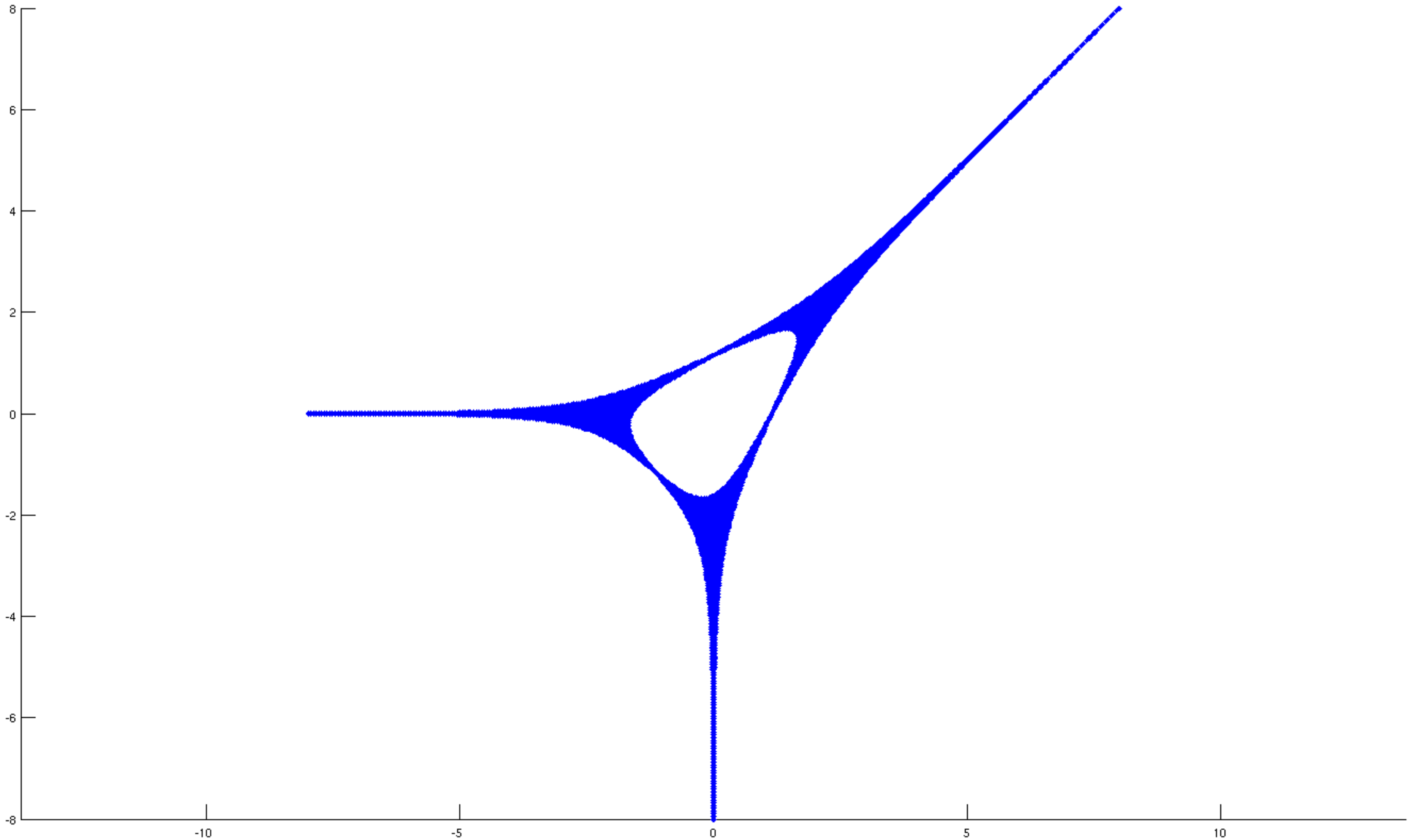


$$f(x_1, x_2) = 1 + x_1 + x_2 + x_1 x_2$$

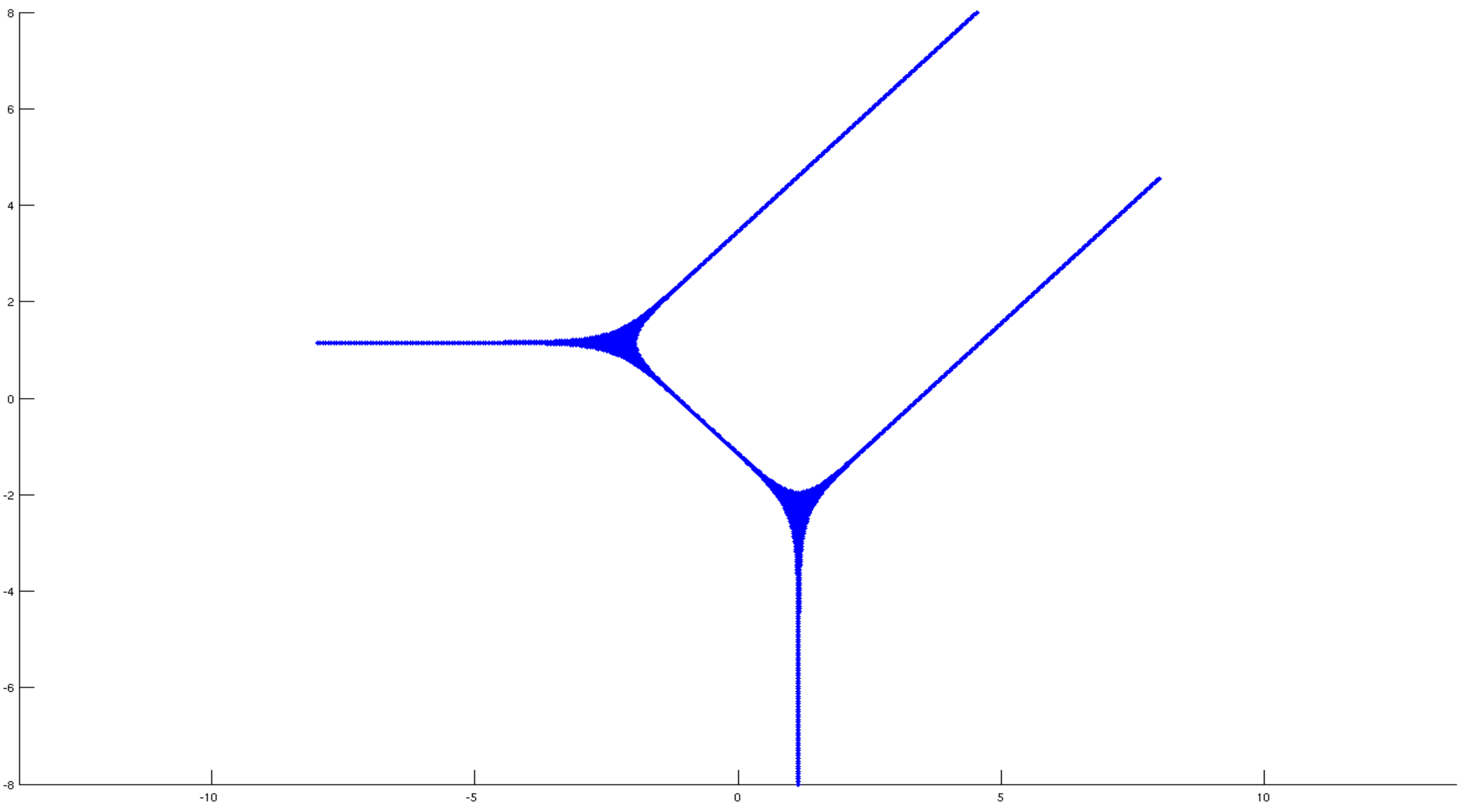
$$= (1 + x_1)(1 + x_2)$$



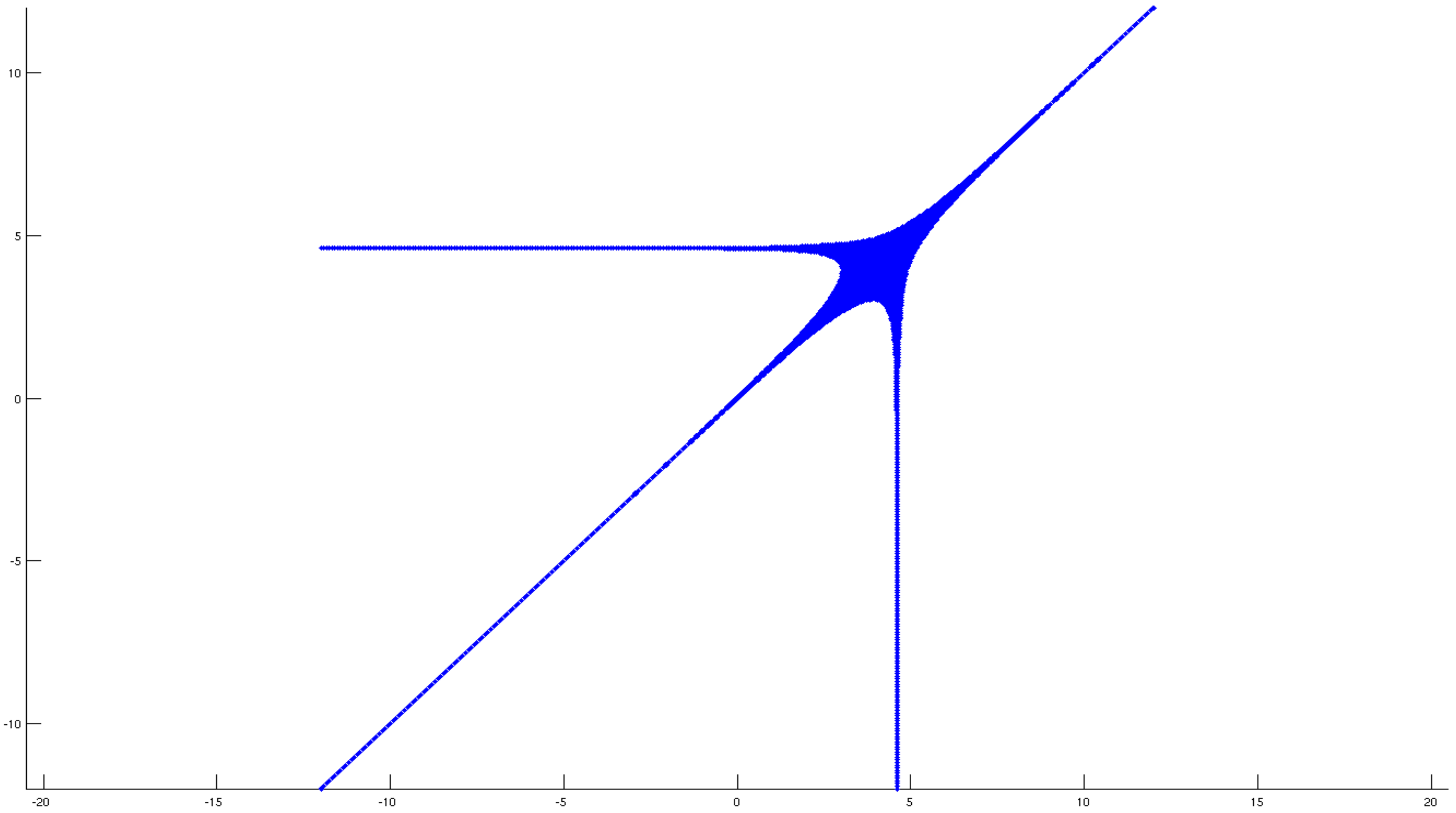
$$f(x_1, x_2) = x_1^3 + x_2^3 + 10x_1x_2 + 1$$



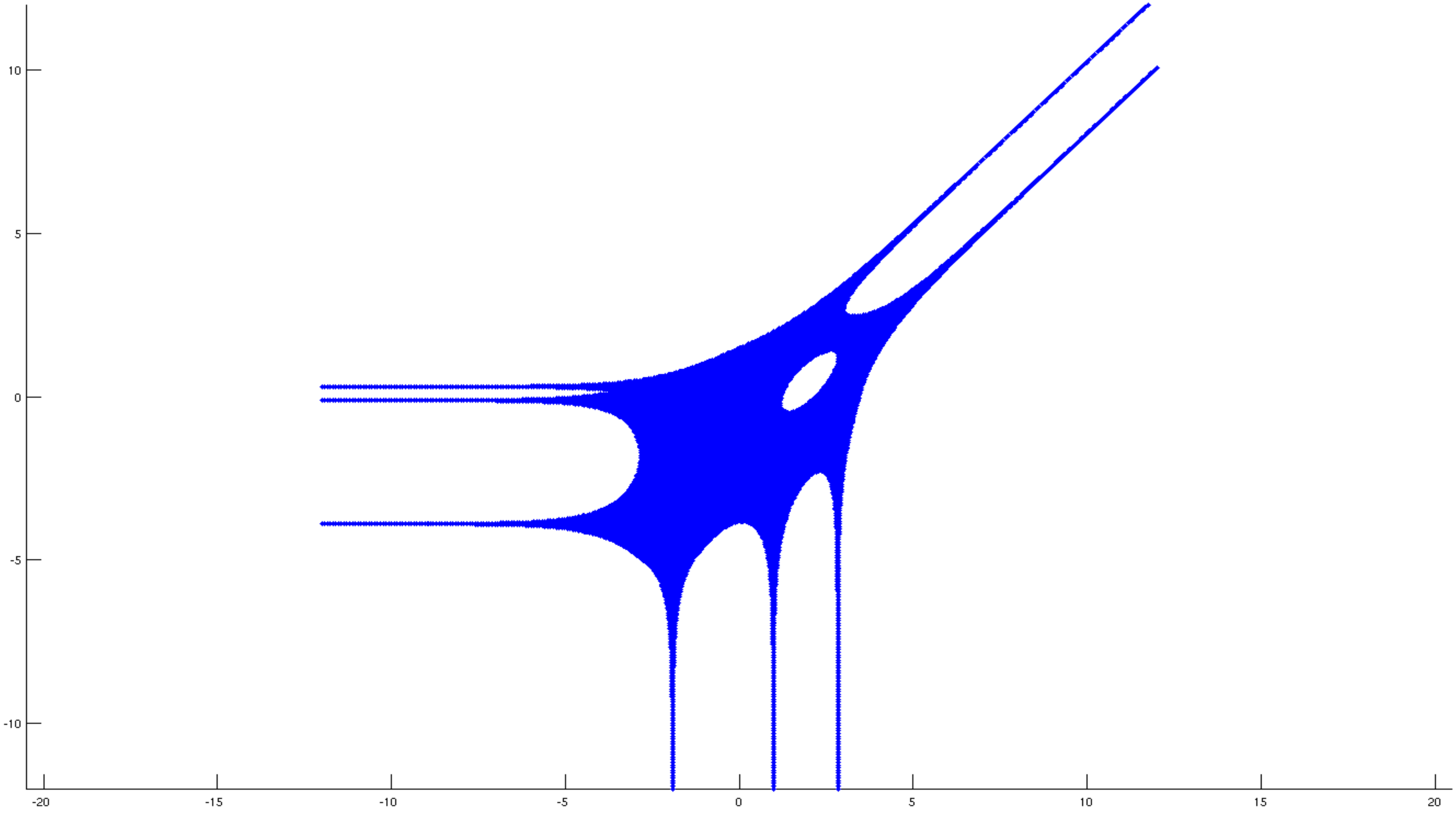
$$f(x_1, x_2) = x_1^4 + x_2^4 + 1000x_1^2x_2^2 + 100$$



$$f(x_1, x_2) = x_1^2 + x_2^2 + 100x_1 + 100x_2$$



$$f(x_1,x_2) = x_1^4 + x_1^3x_2 + 50x_1^2x_2^2 + 40x_1x_2^3 + 30x_2^4 + 20x_1^3 + 460x_1^2x_2 + 480x_1x_2^2 + 10x_2^3 + 50x_1^2 - 500x_1x_2 + 40x_2^2 + 10x_1 + 50x_2 + 1$$



Liftings

- For $f(x) = \sum_{i=1}^{t} c_i x^{a_i}$ where $a_1, \ldots, a_t \in \mathbb{Z}^2$:

$\mathrm{Newt}(f)$ is the convex hull of $\{a_1, \ldots, a_t\}$

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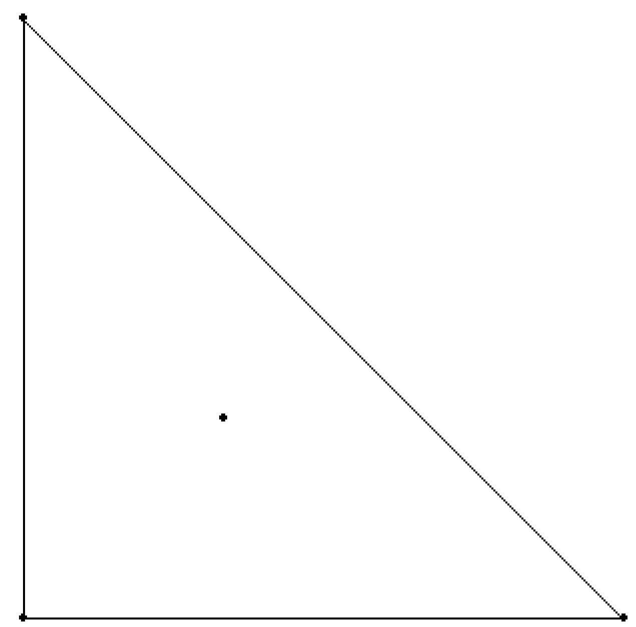
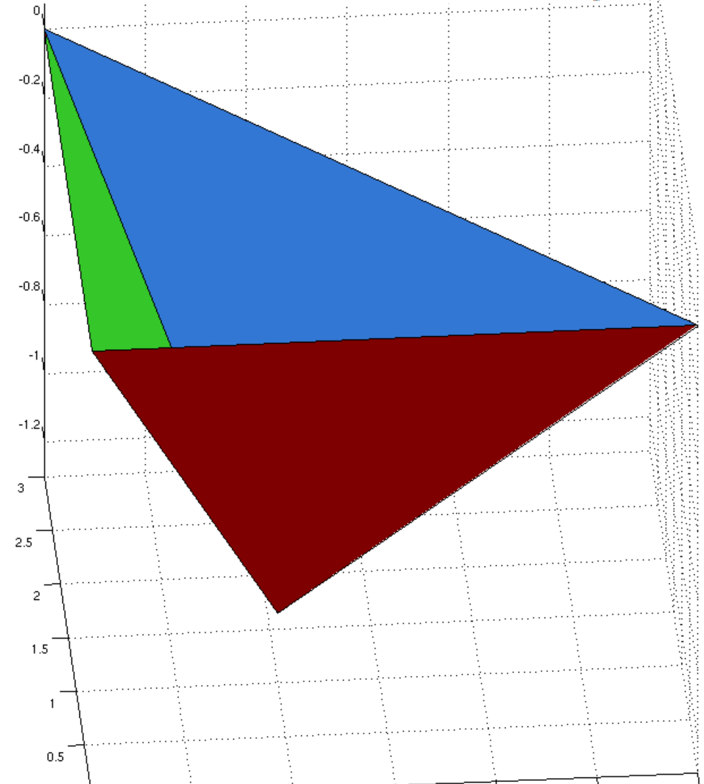


Liftings

- For $f(x)=\sum_{i=1}^{t} c_i x^{a_i}$ where $a_1,\ldots,a_t \in \mathbb{Z}^2$:

$\mathrm{ArchNewt}(f)$ is the convex hull of $\{(a_i, -\log|c_i|)\}_{i\in\{1,\ldots,t\}}$

$$f(x_1,x_2) = x_1^3 + x_2^3 + 10x_1x_2 + 1$$

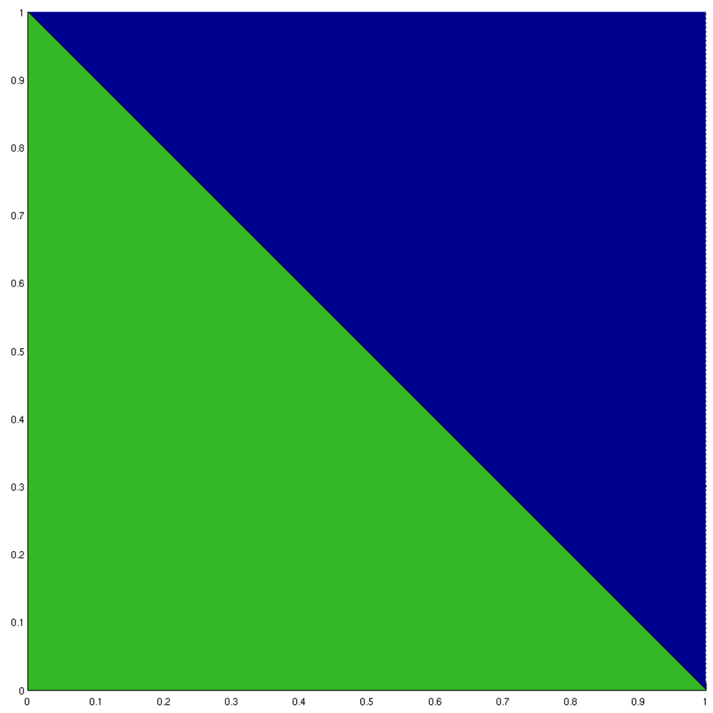


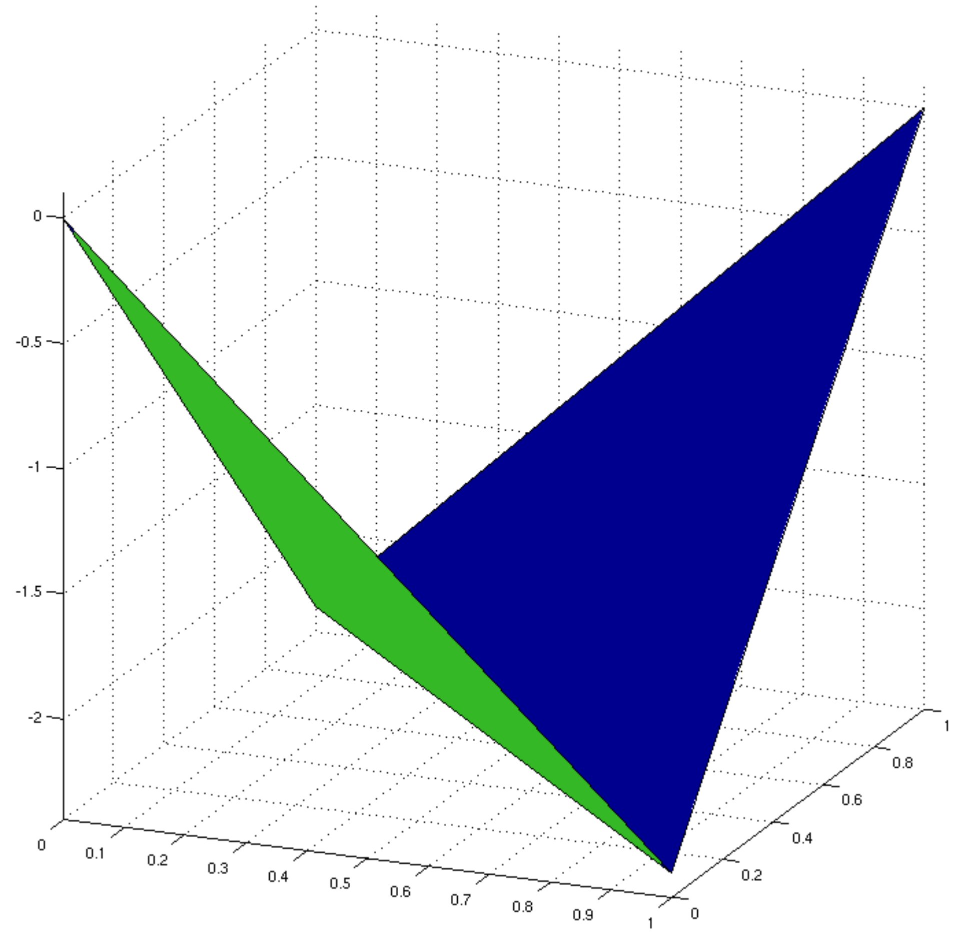
Avendaño's Theorem

- Let $\mathrm{Trop}(f)$ denote the intersection of the inner normal fan of $\mathrm{ArchNewt}(f)$ with the hyperplane $\{x_{n+1} = 1\}$
- The Theorem:

$$\Delta(-\mathrm{Amoeba}(f), \mathrm{Trop}(f)) \leq \log(t-1)$$

An Example: $f(x_1, x_2) = 1 + 10x_1 + 10x_2 + x_1x_2$





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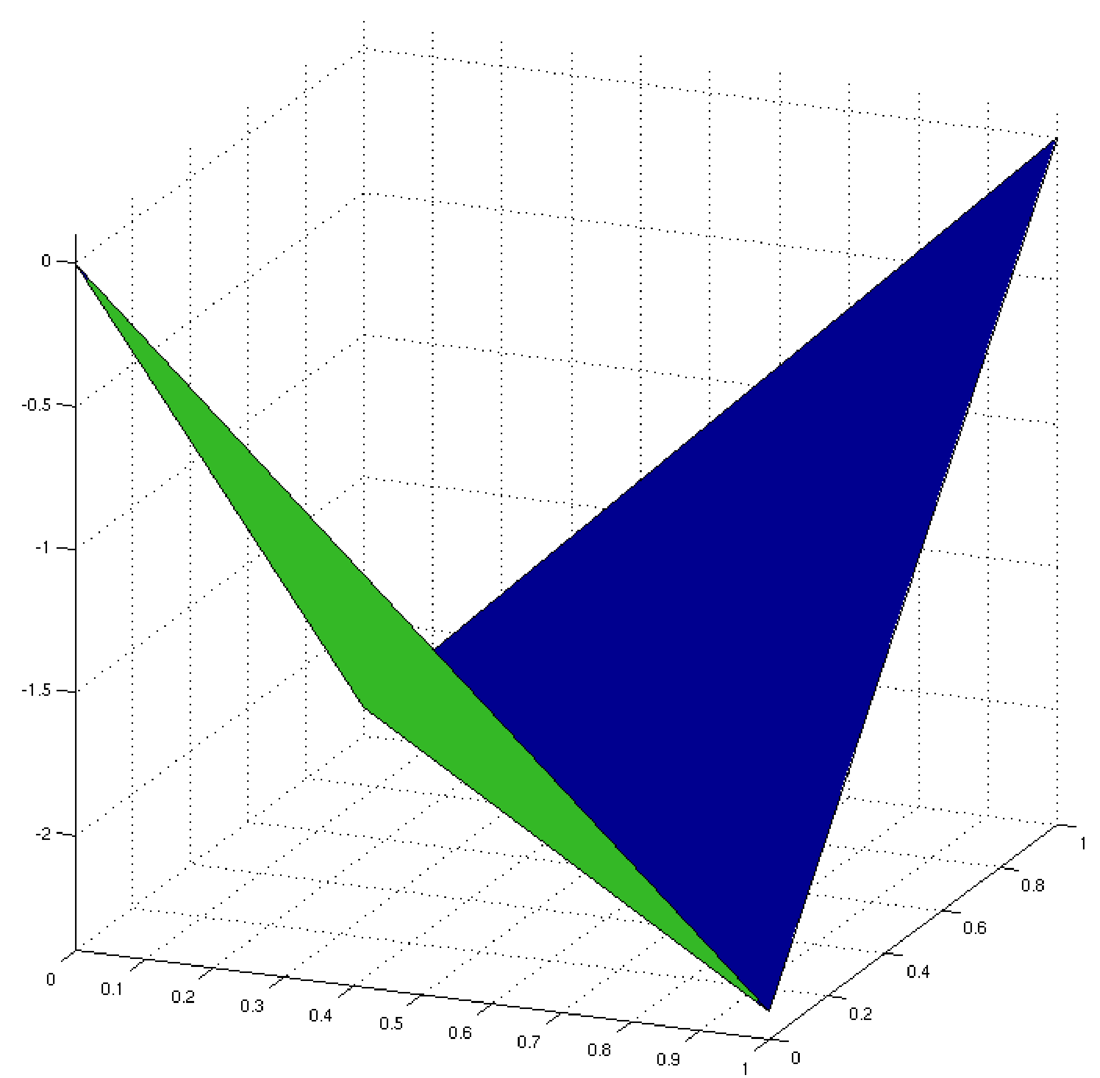

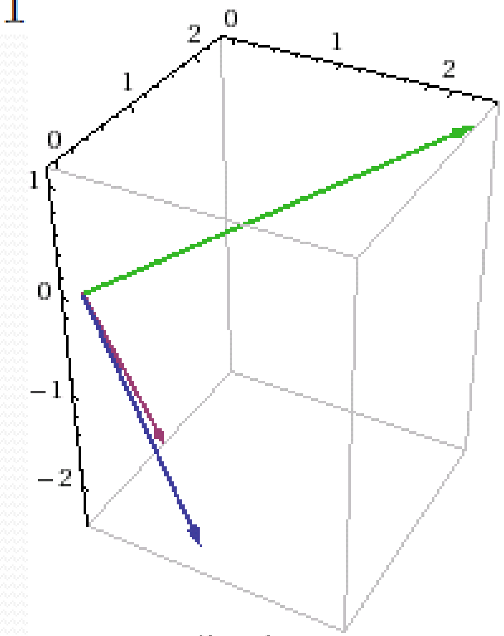

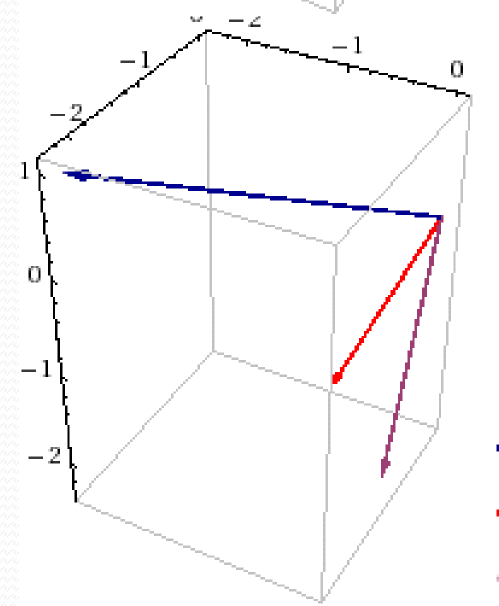

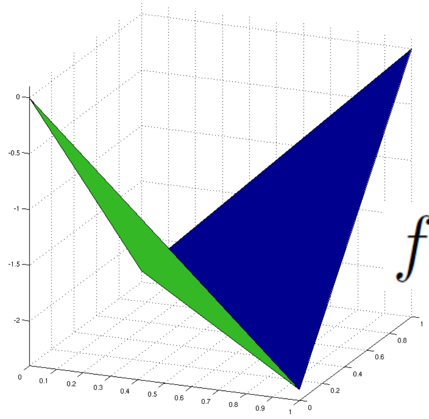

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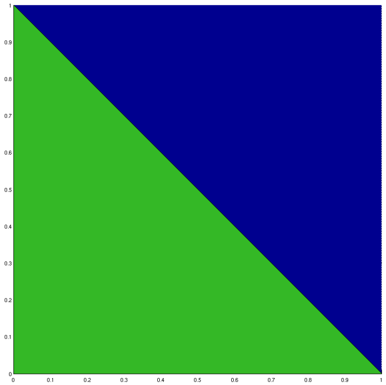

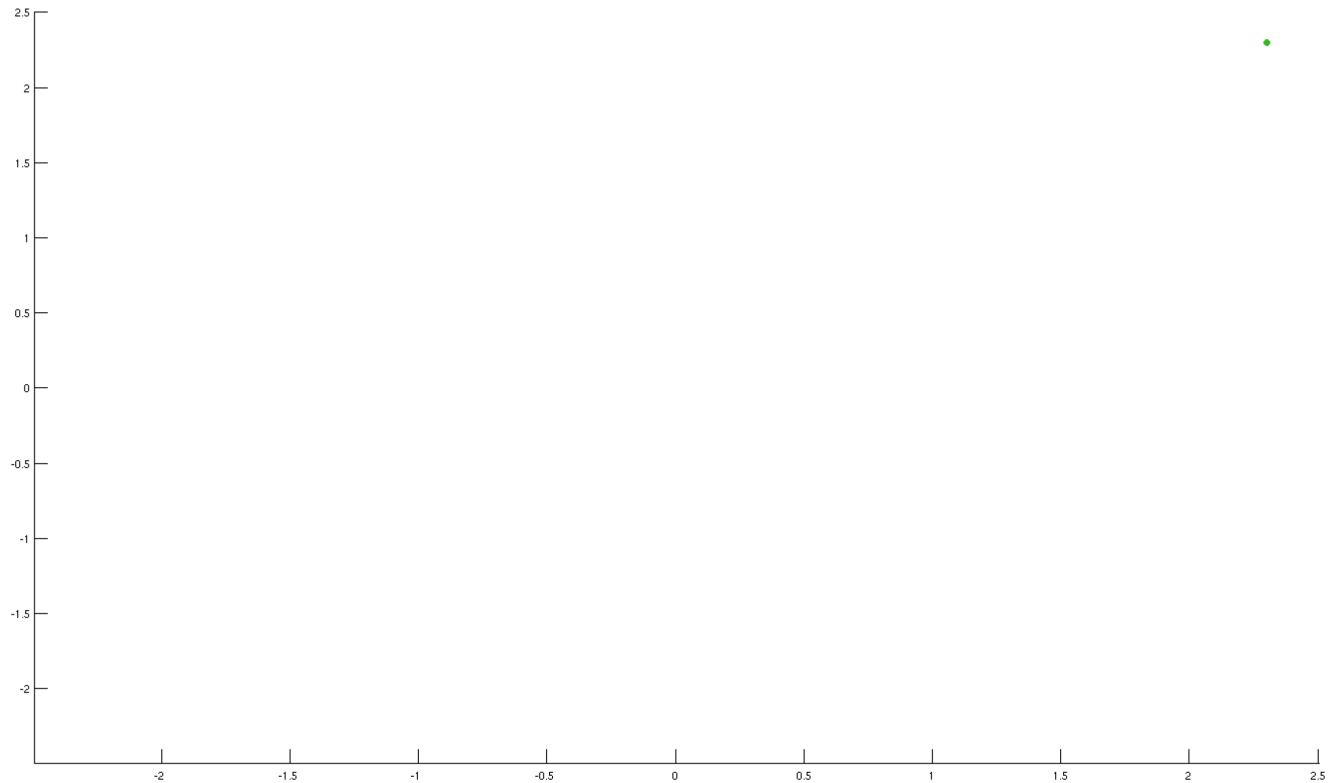

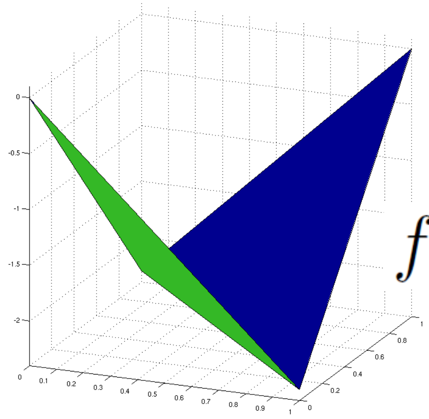

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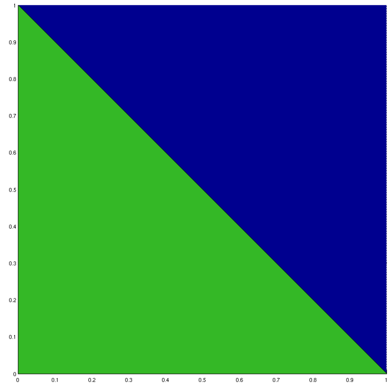

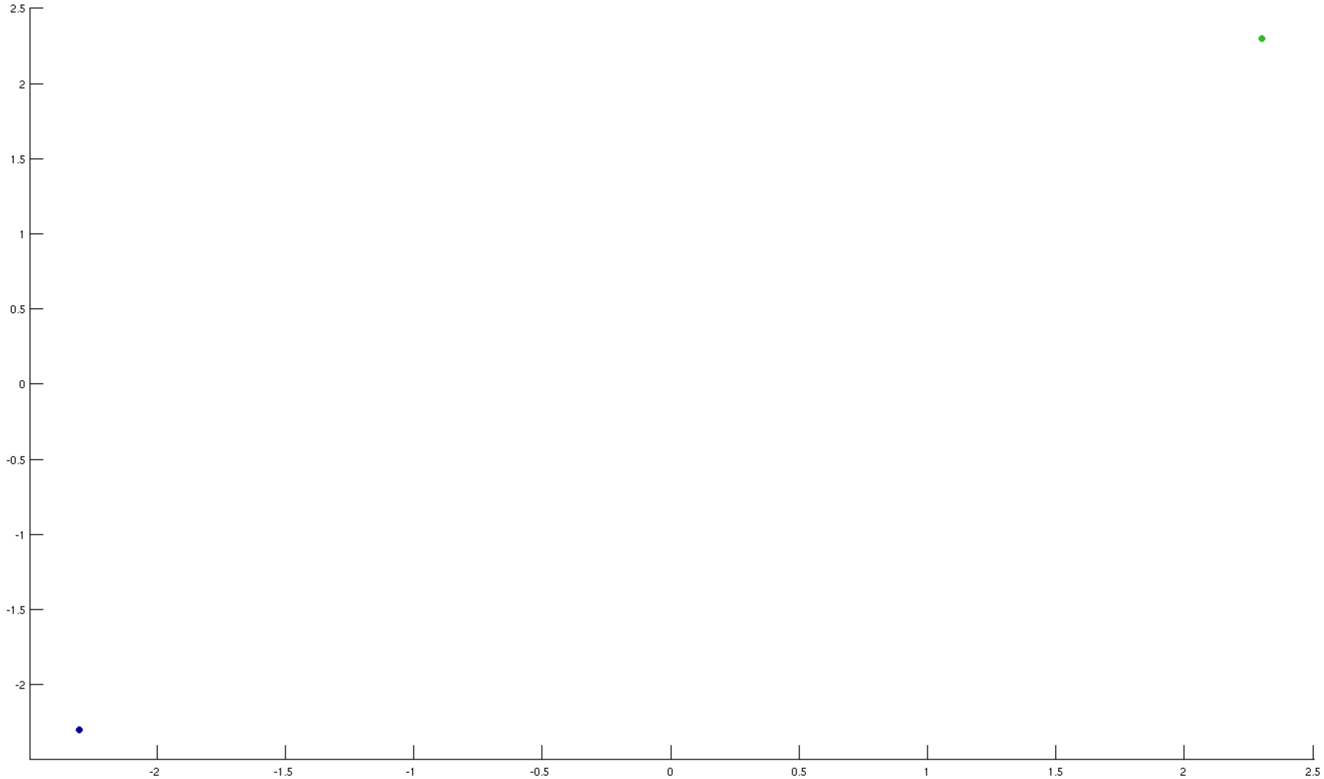

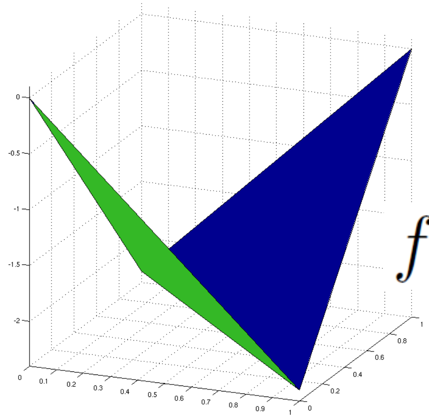

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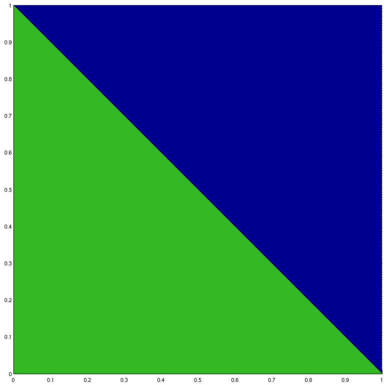

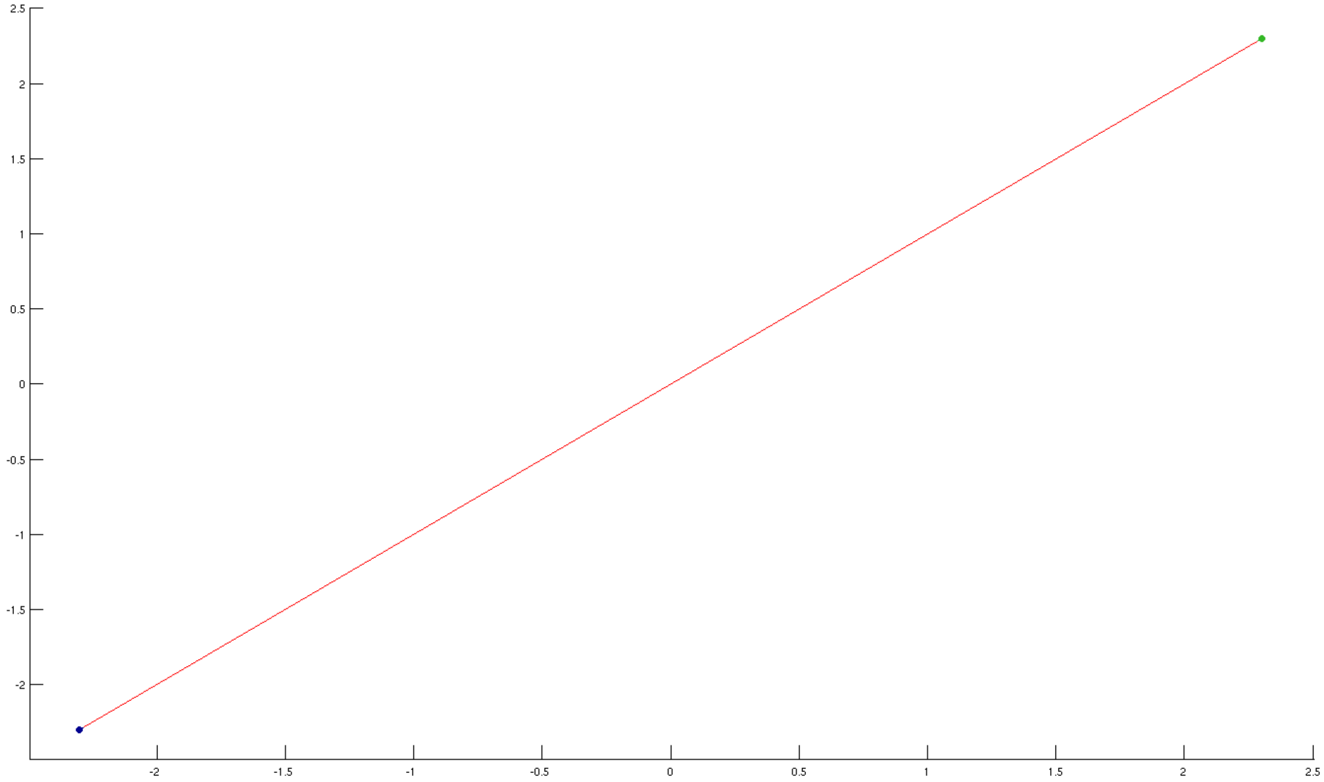

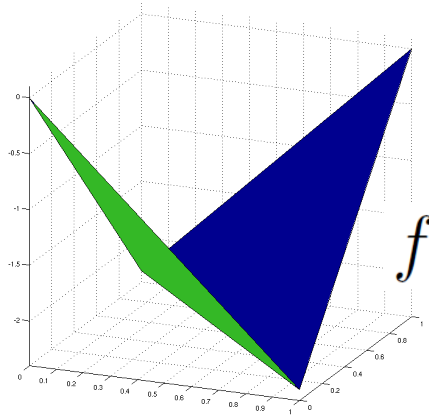

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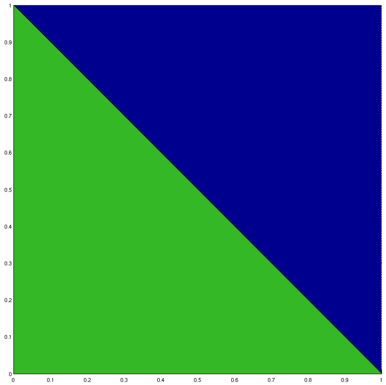

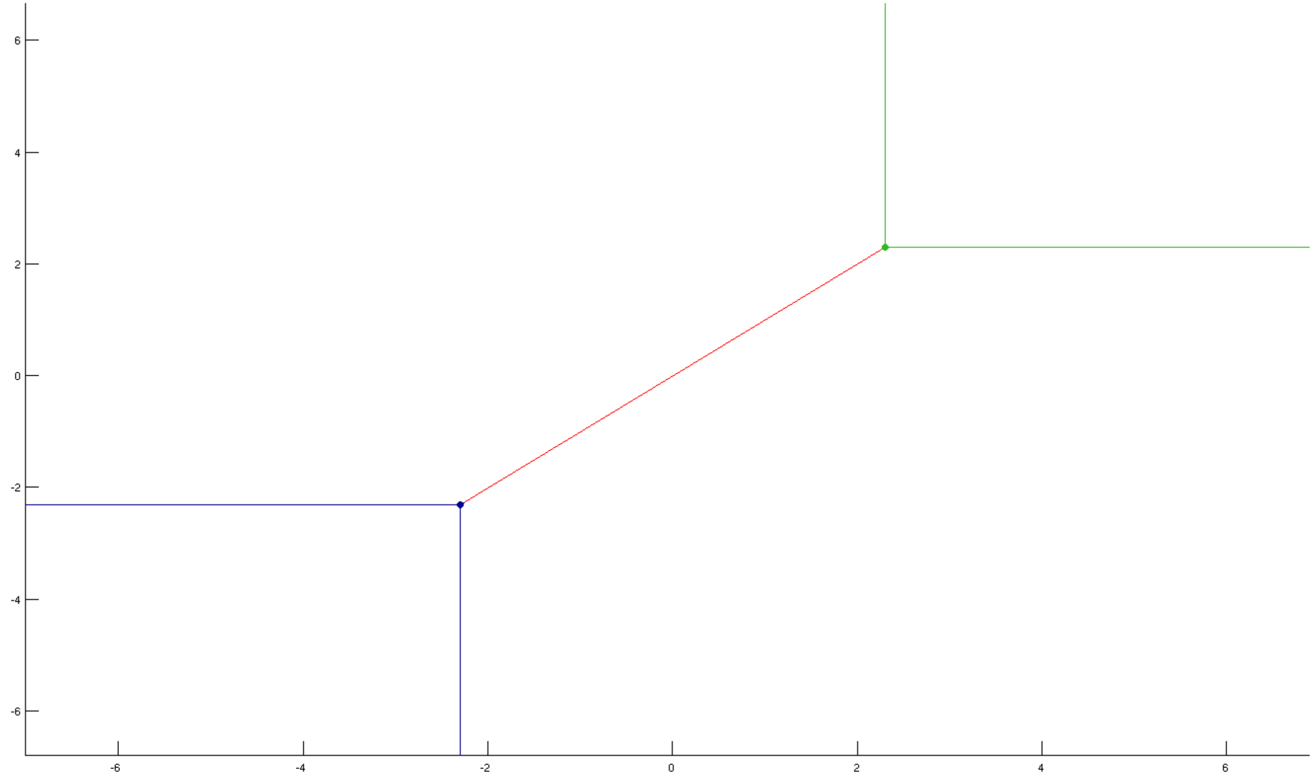

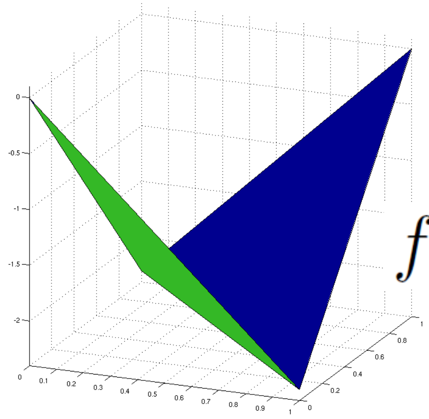

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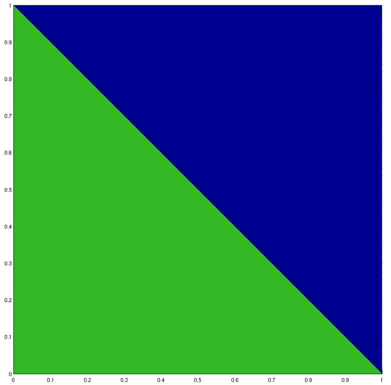

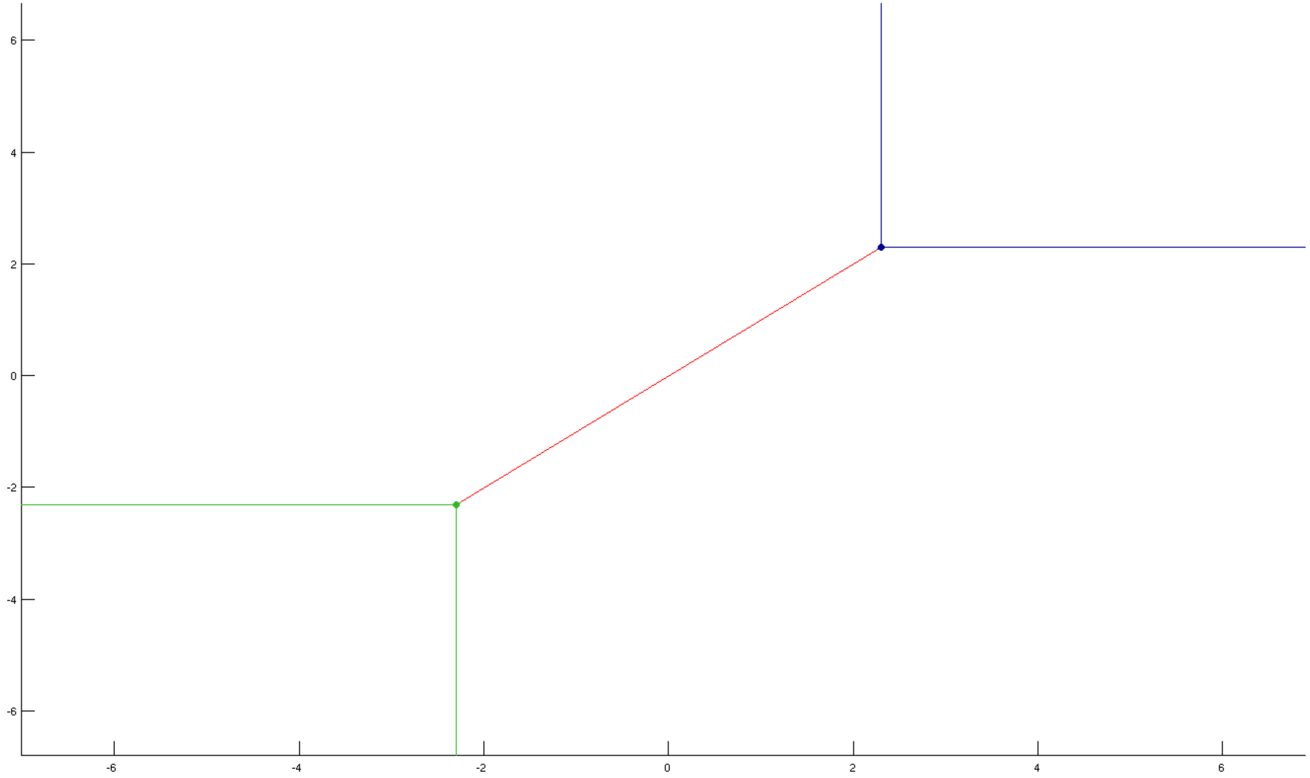

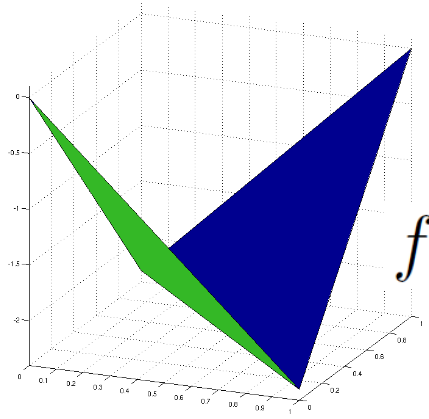

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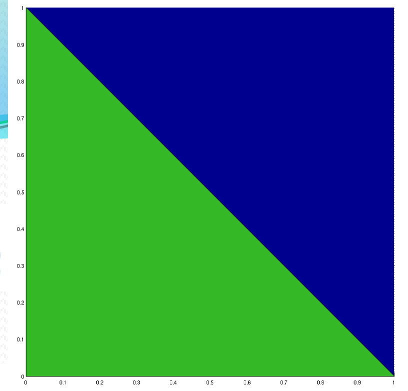

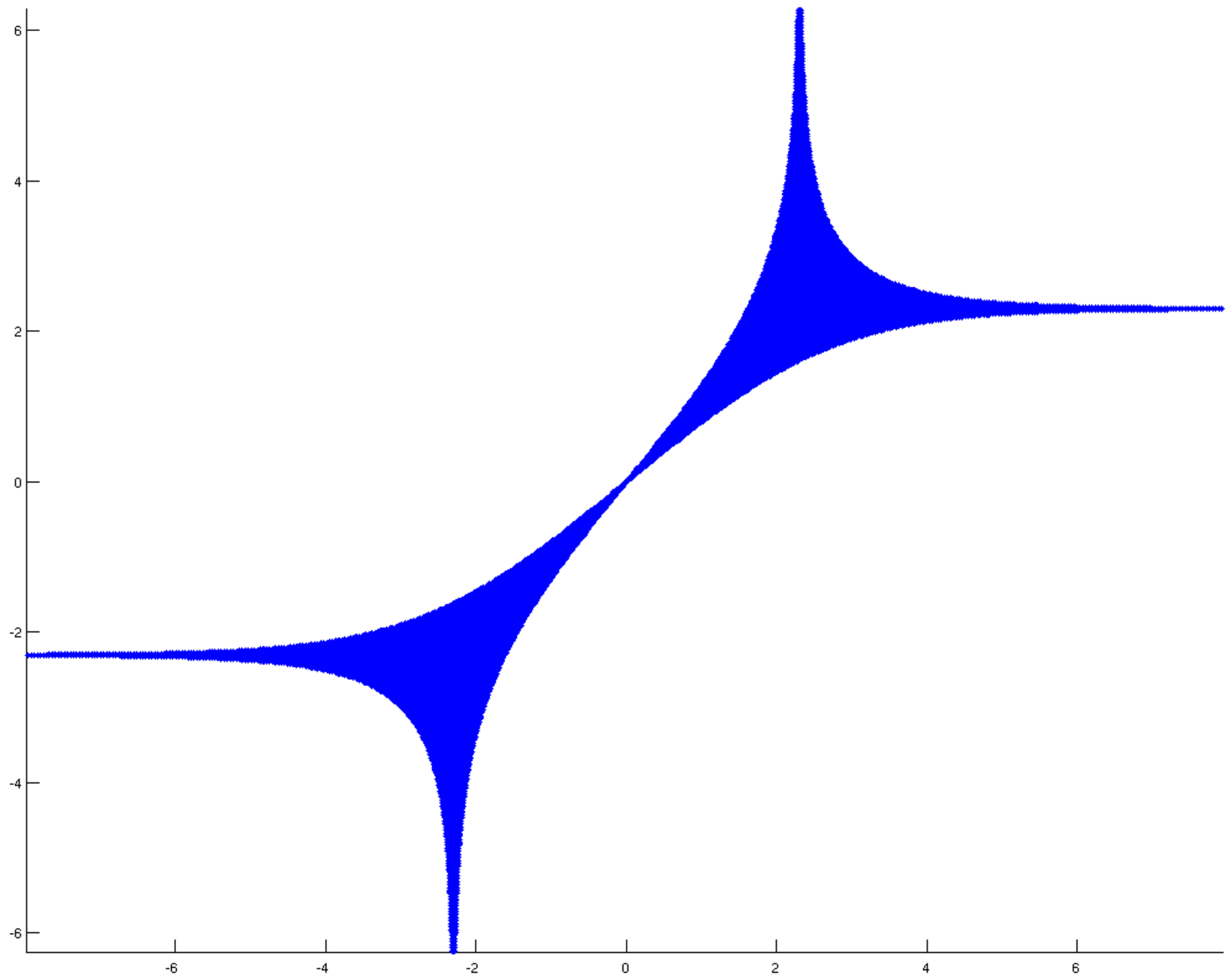

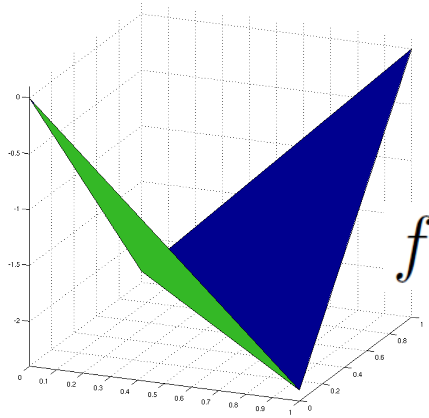

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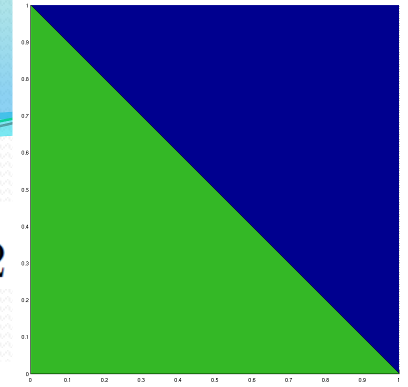
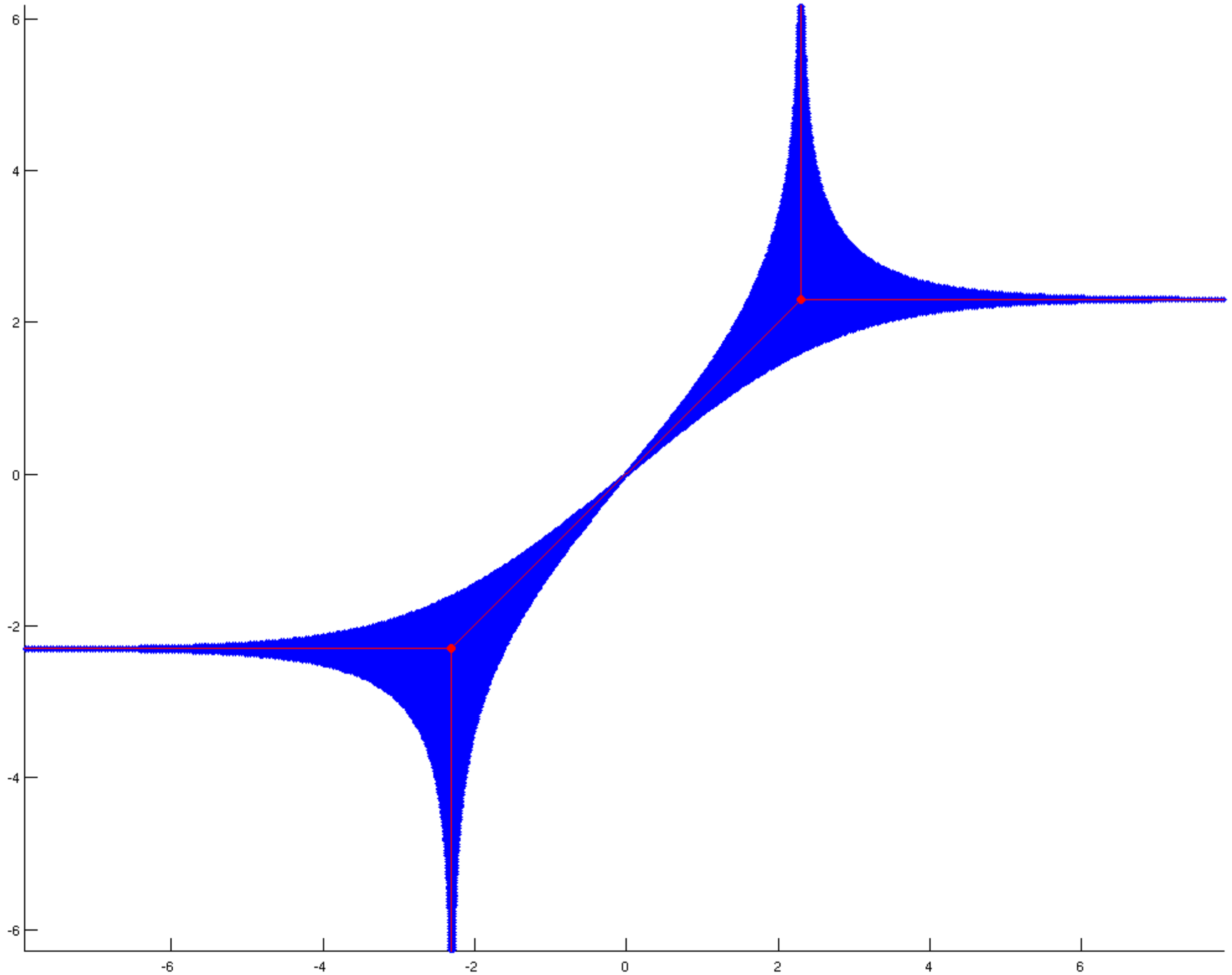
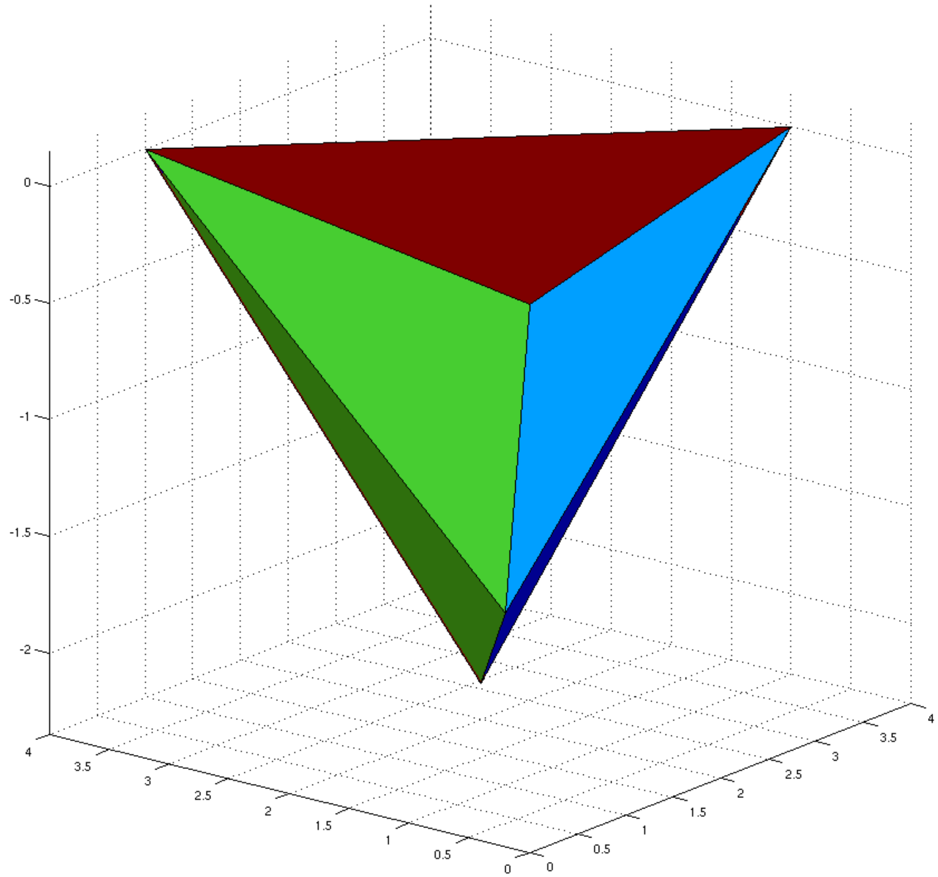
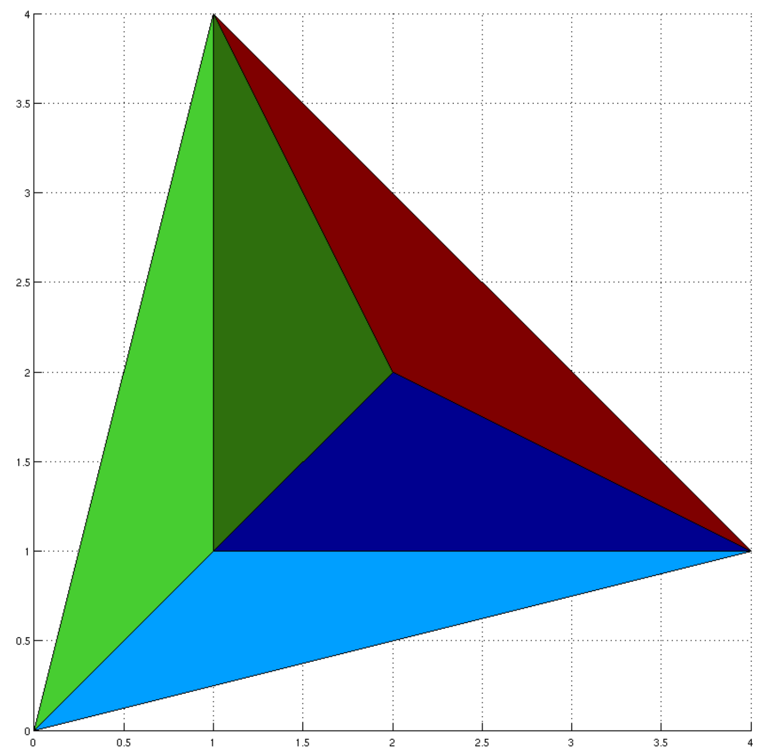




$$f(x_1, x_2) = x_1^4 x_2 + x_1 x_2^4 - 9x_1^2 x_2^2 + 5x_1 x_2 + 1$$





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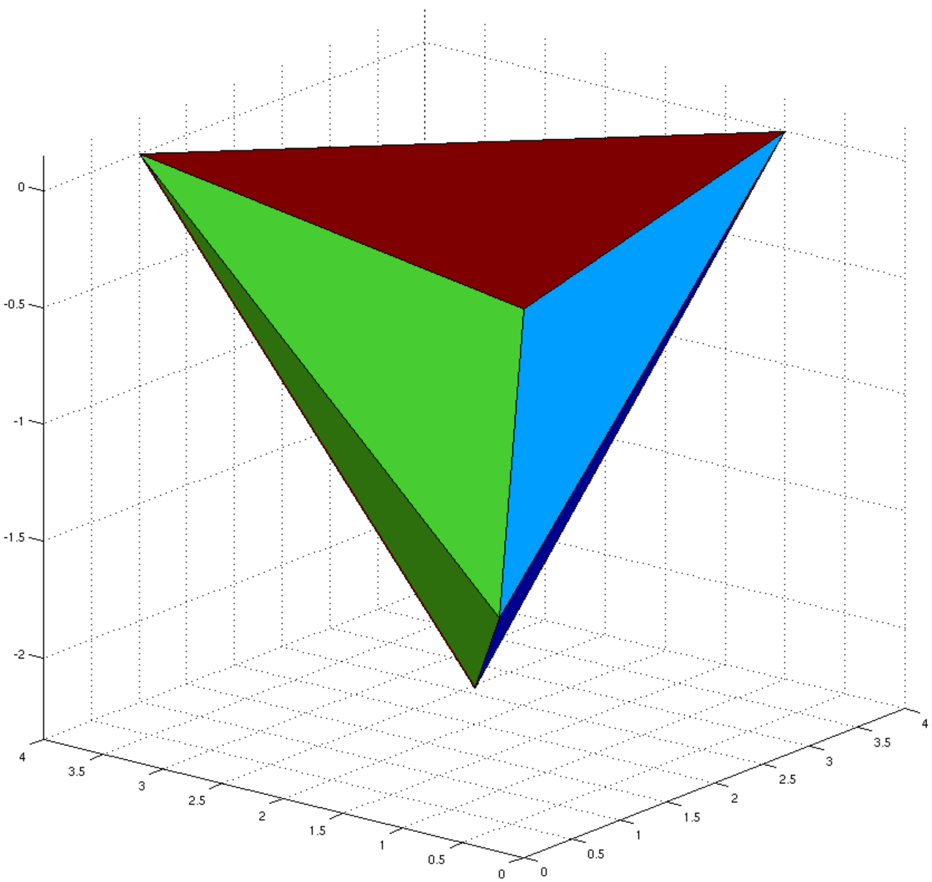

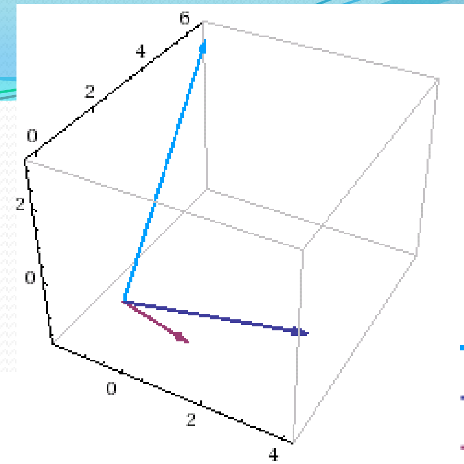

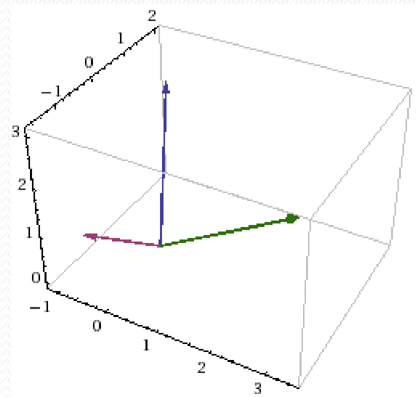

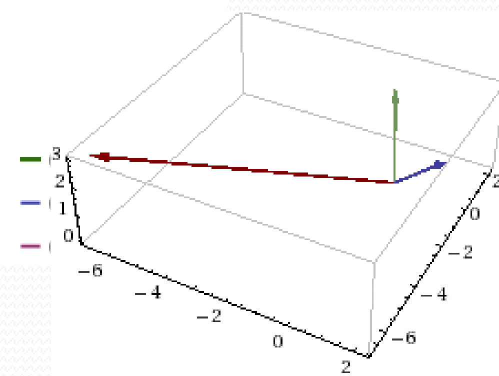

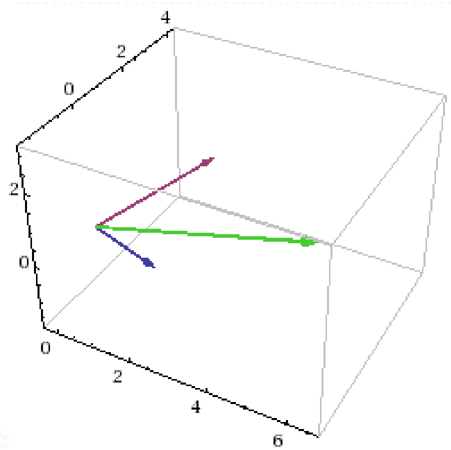

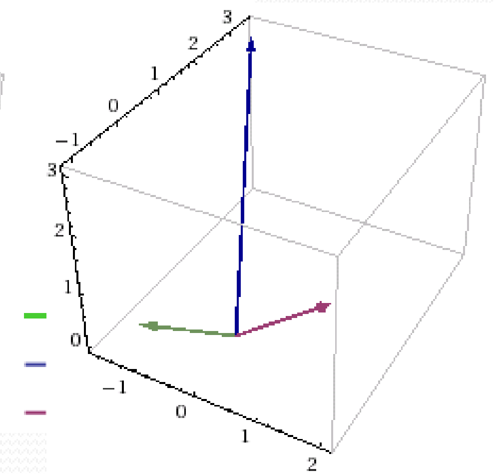

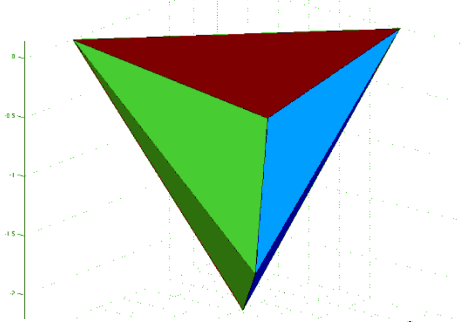

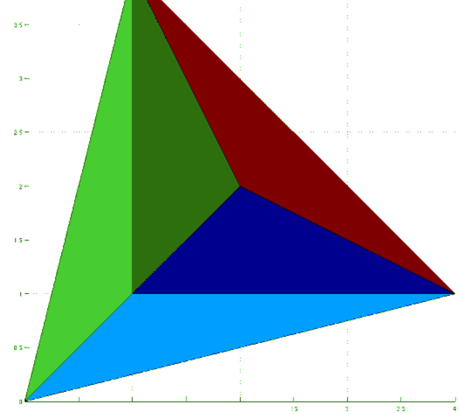

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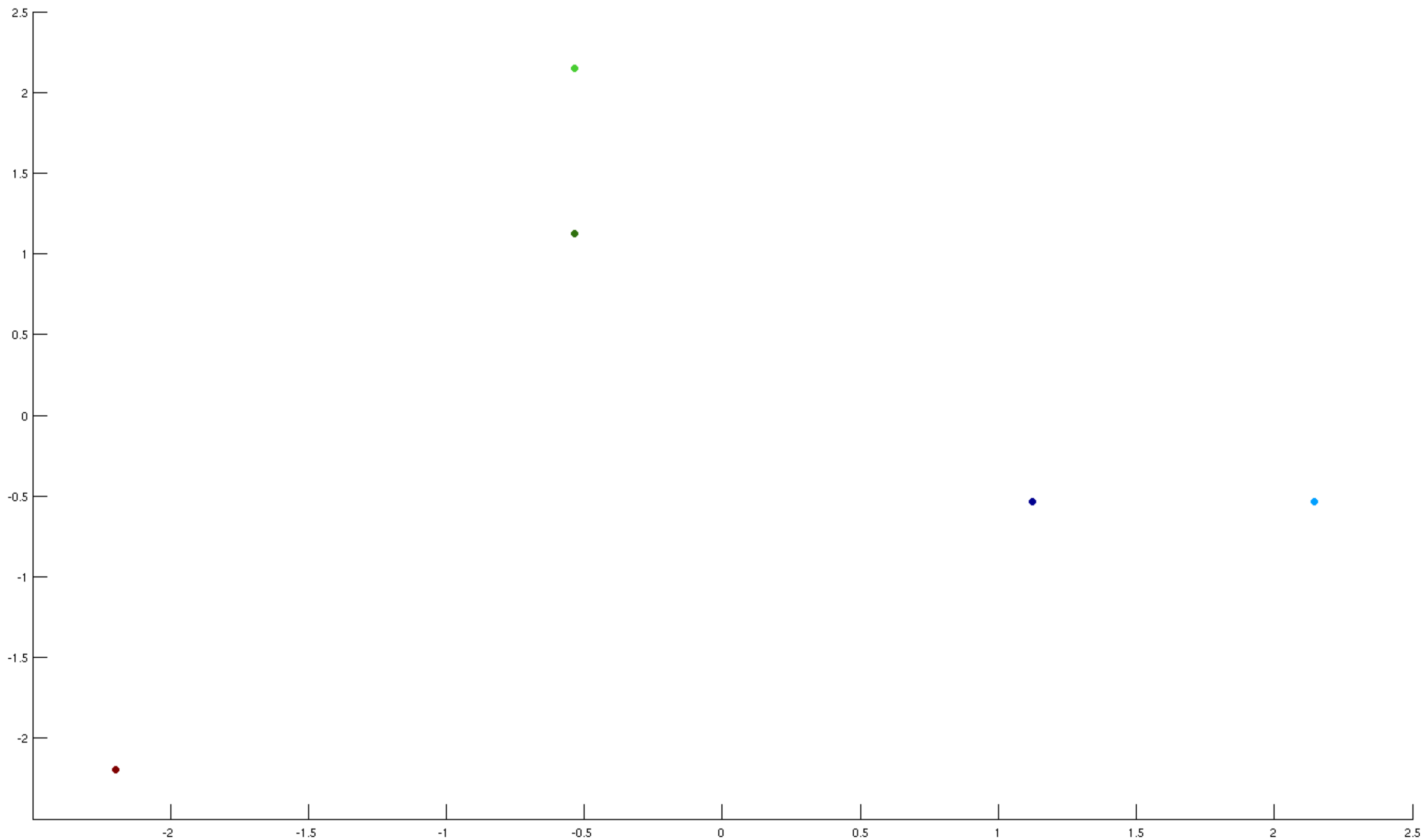

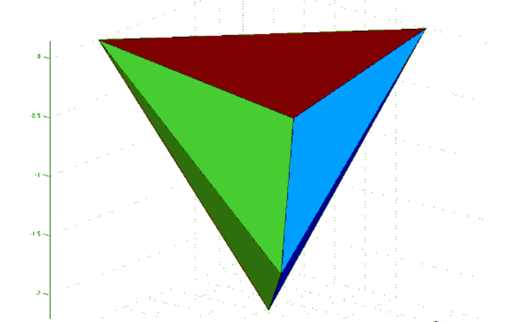

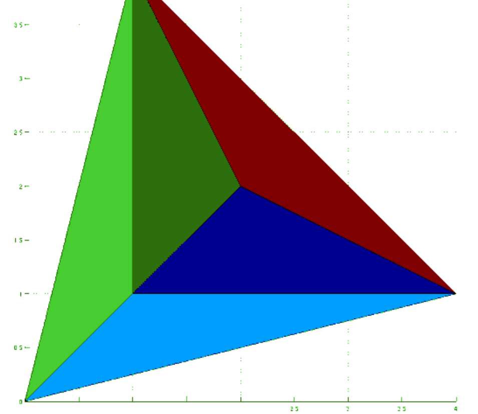

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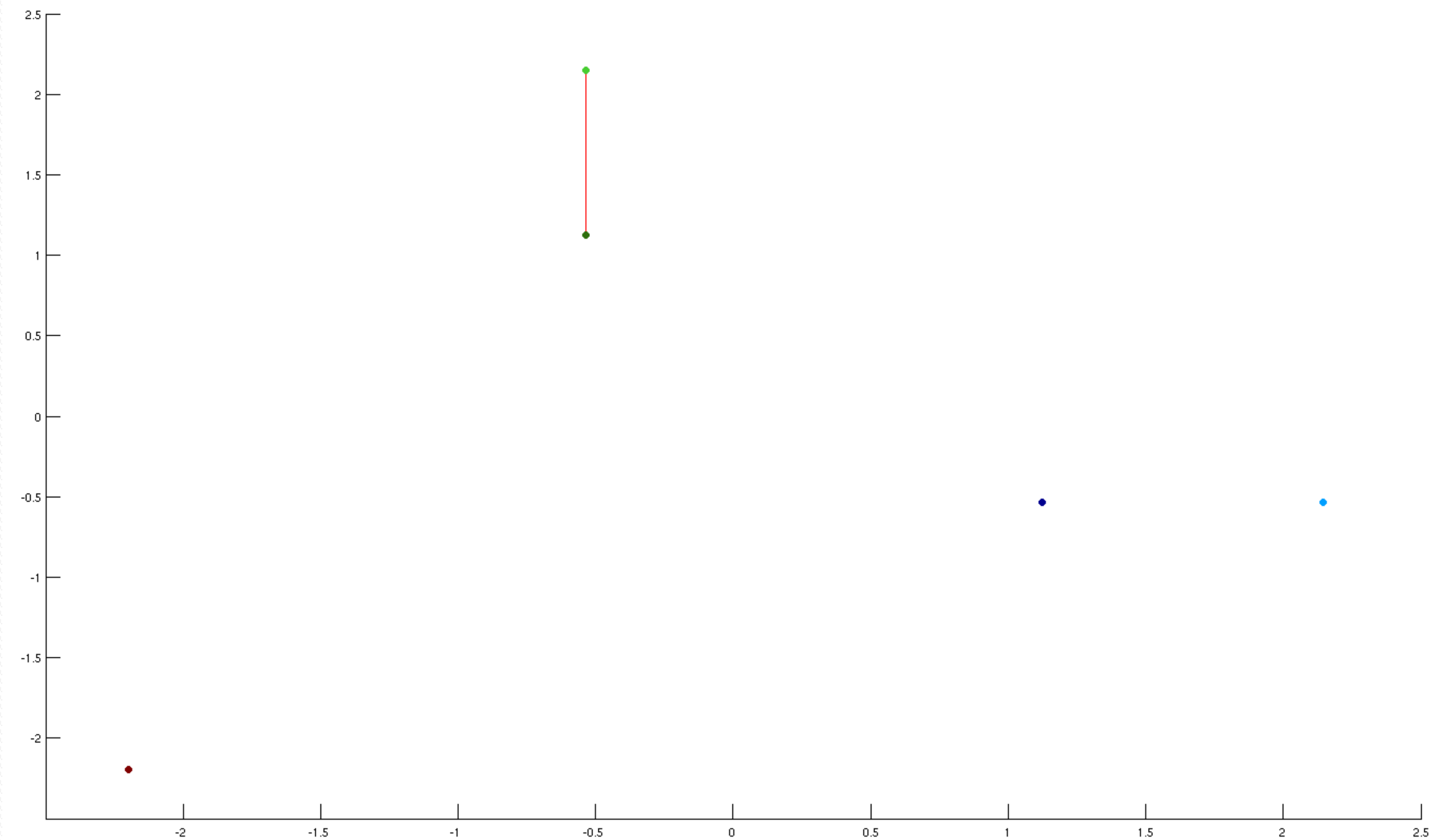

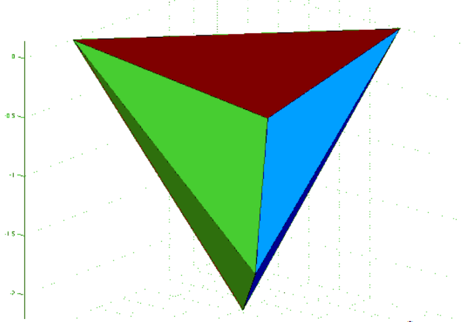

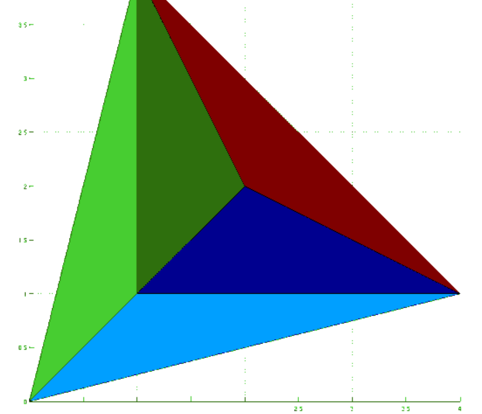

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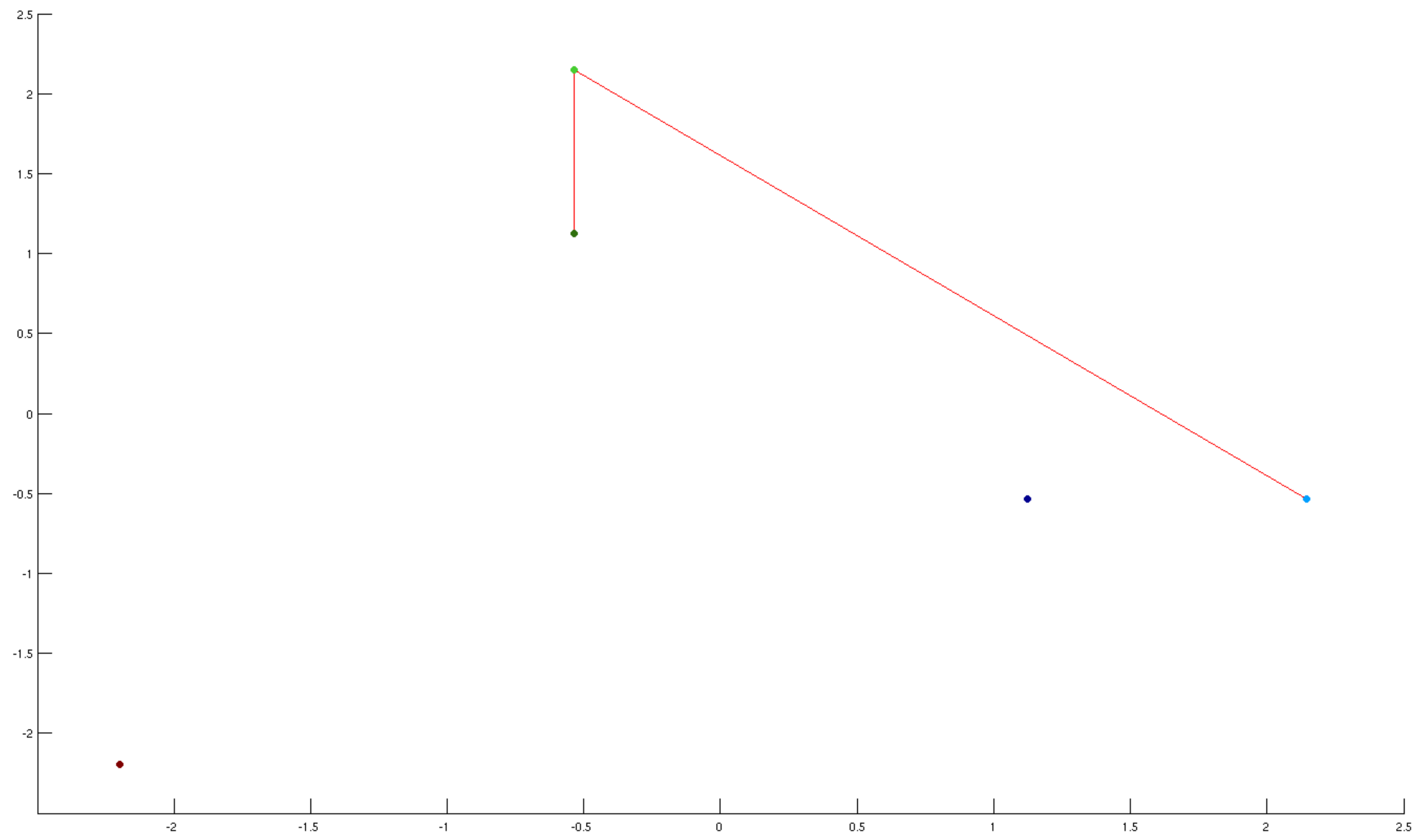

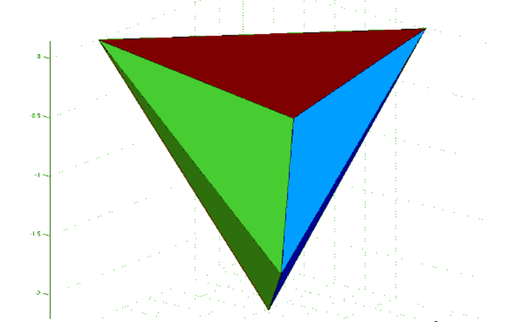

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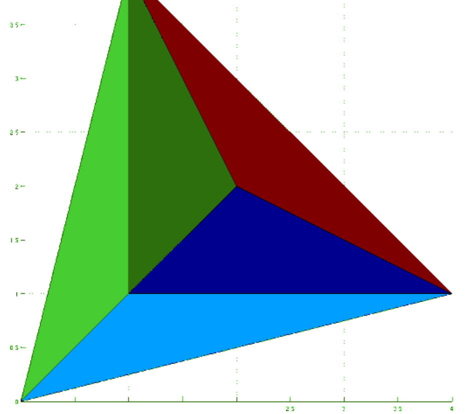

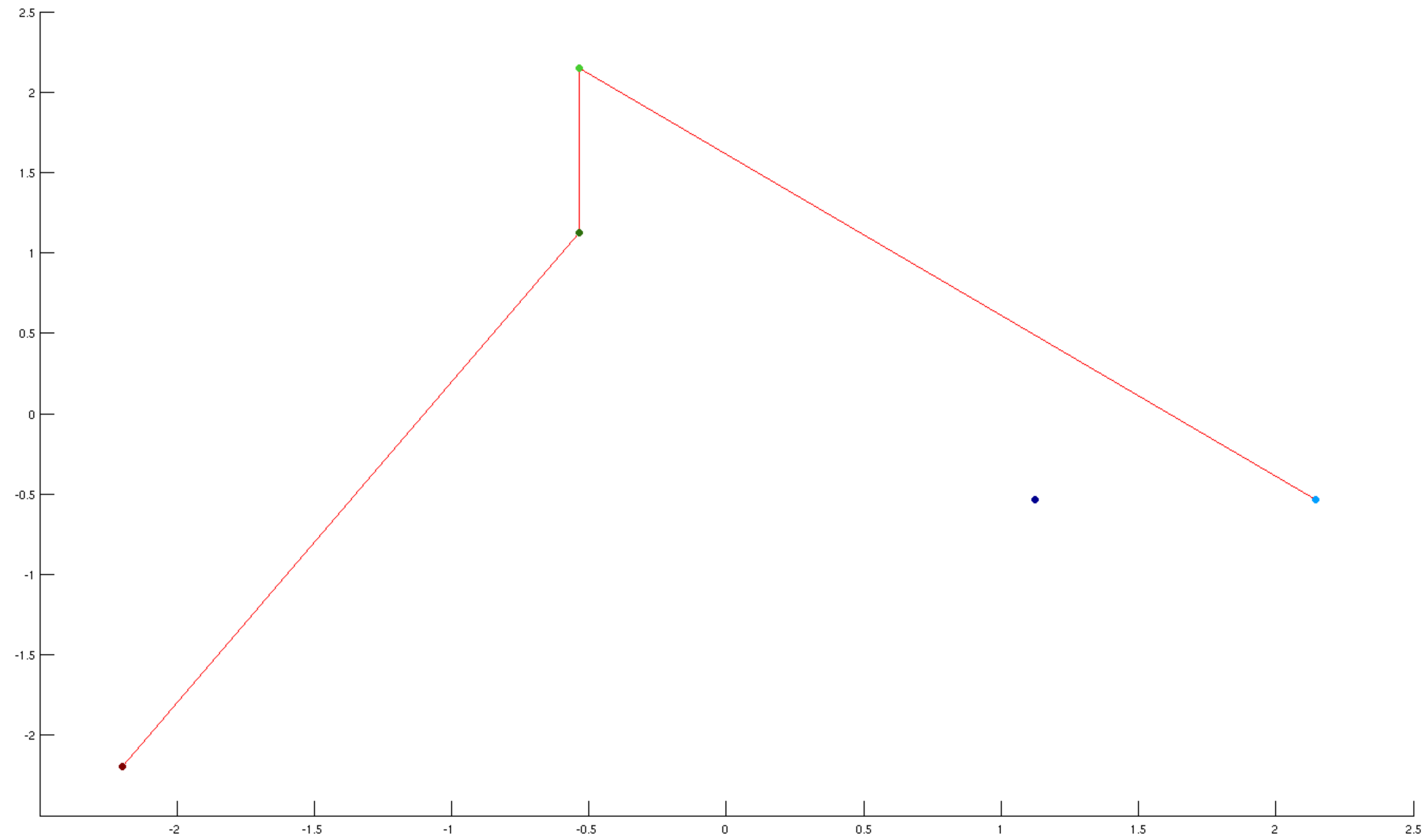

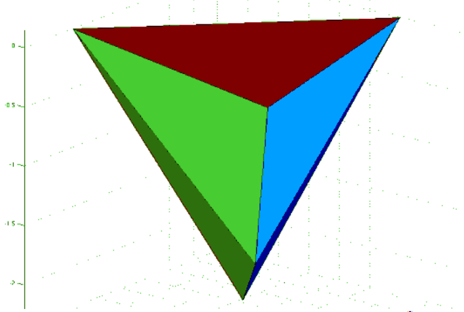

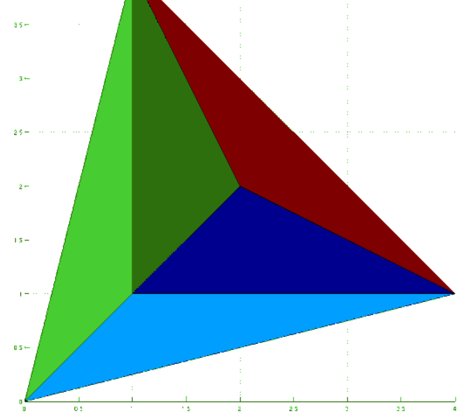

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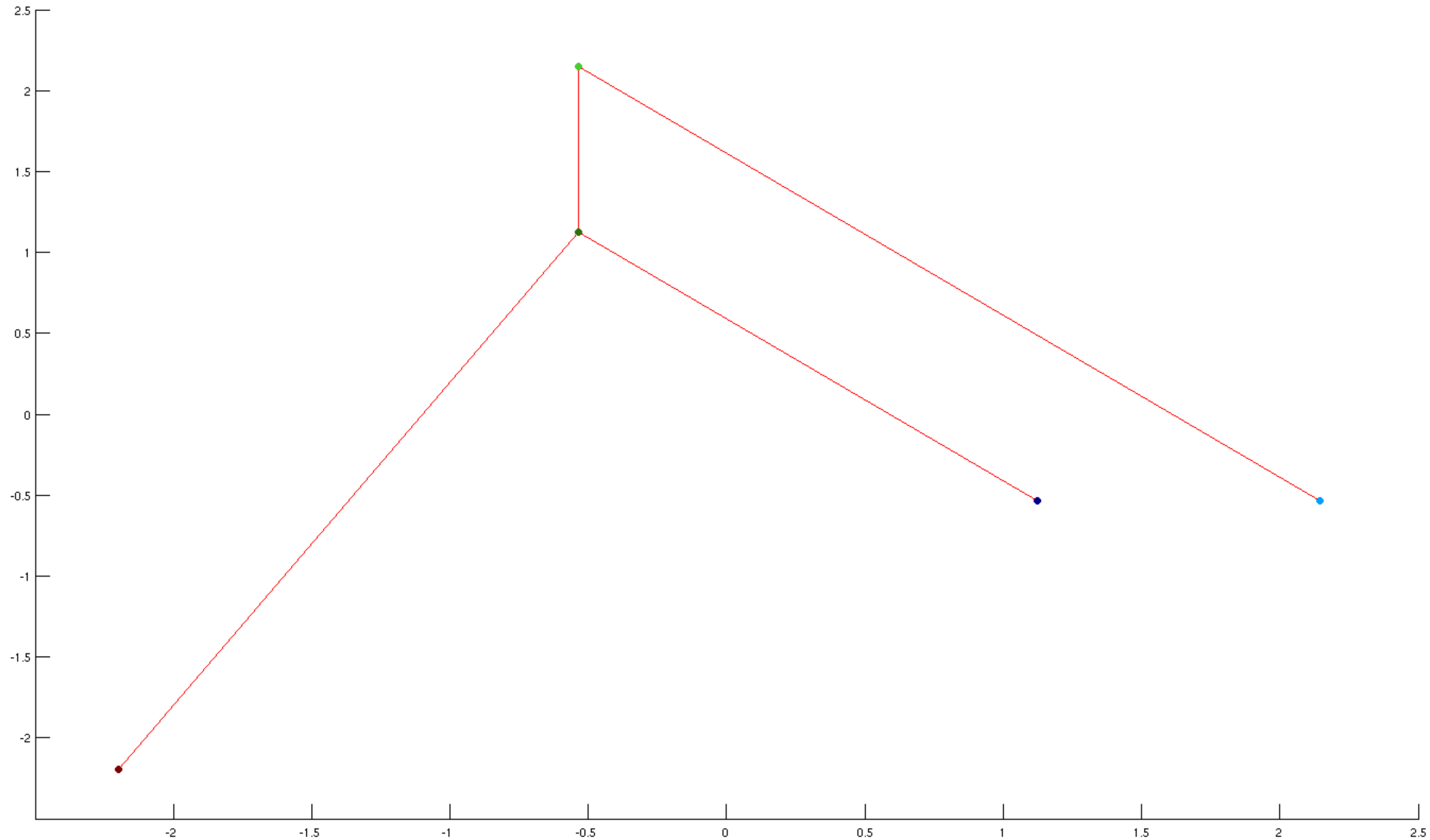

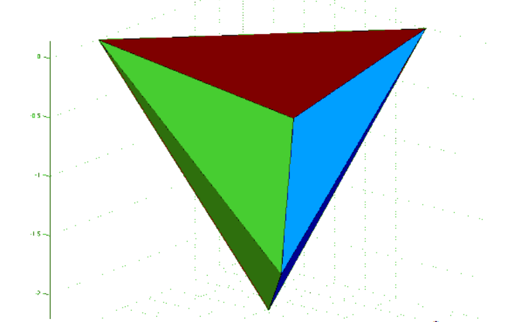

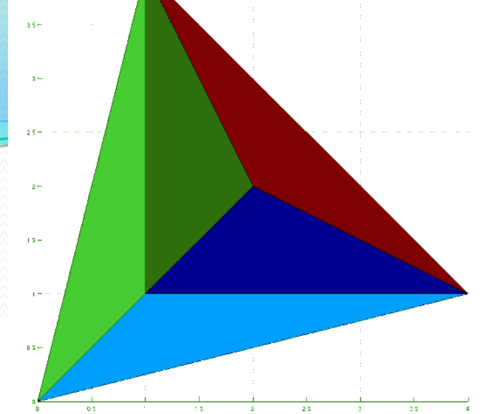

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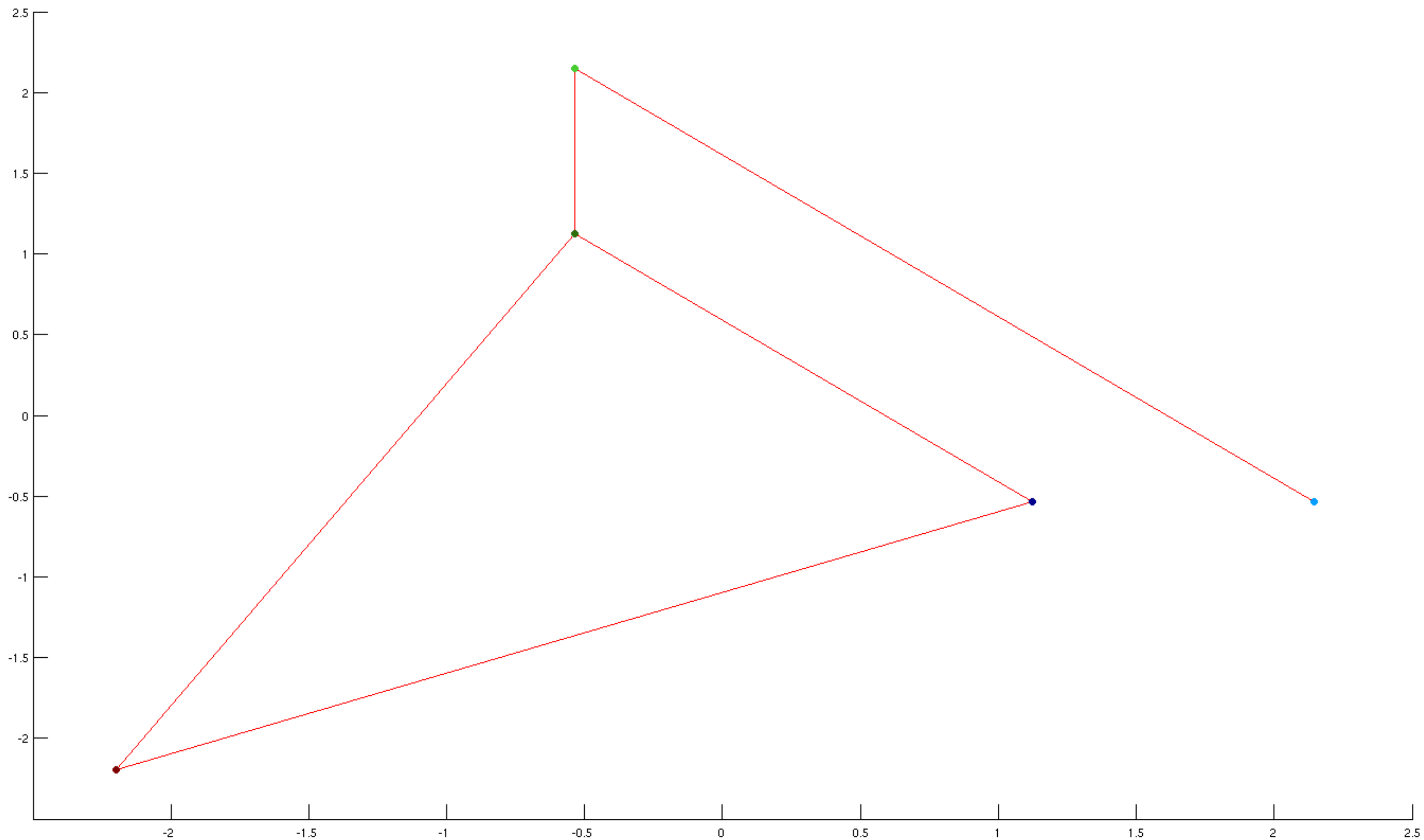

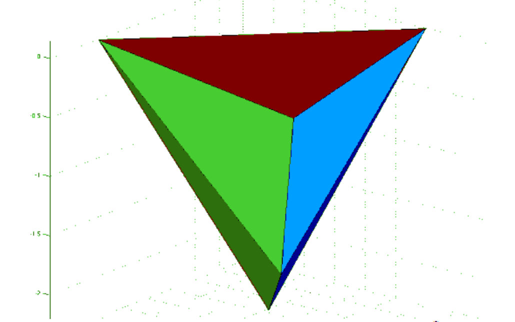

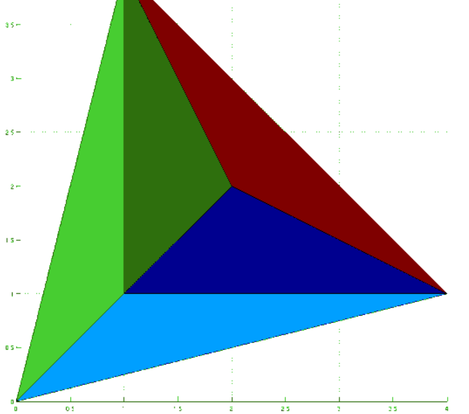

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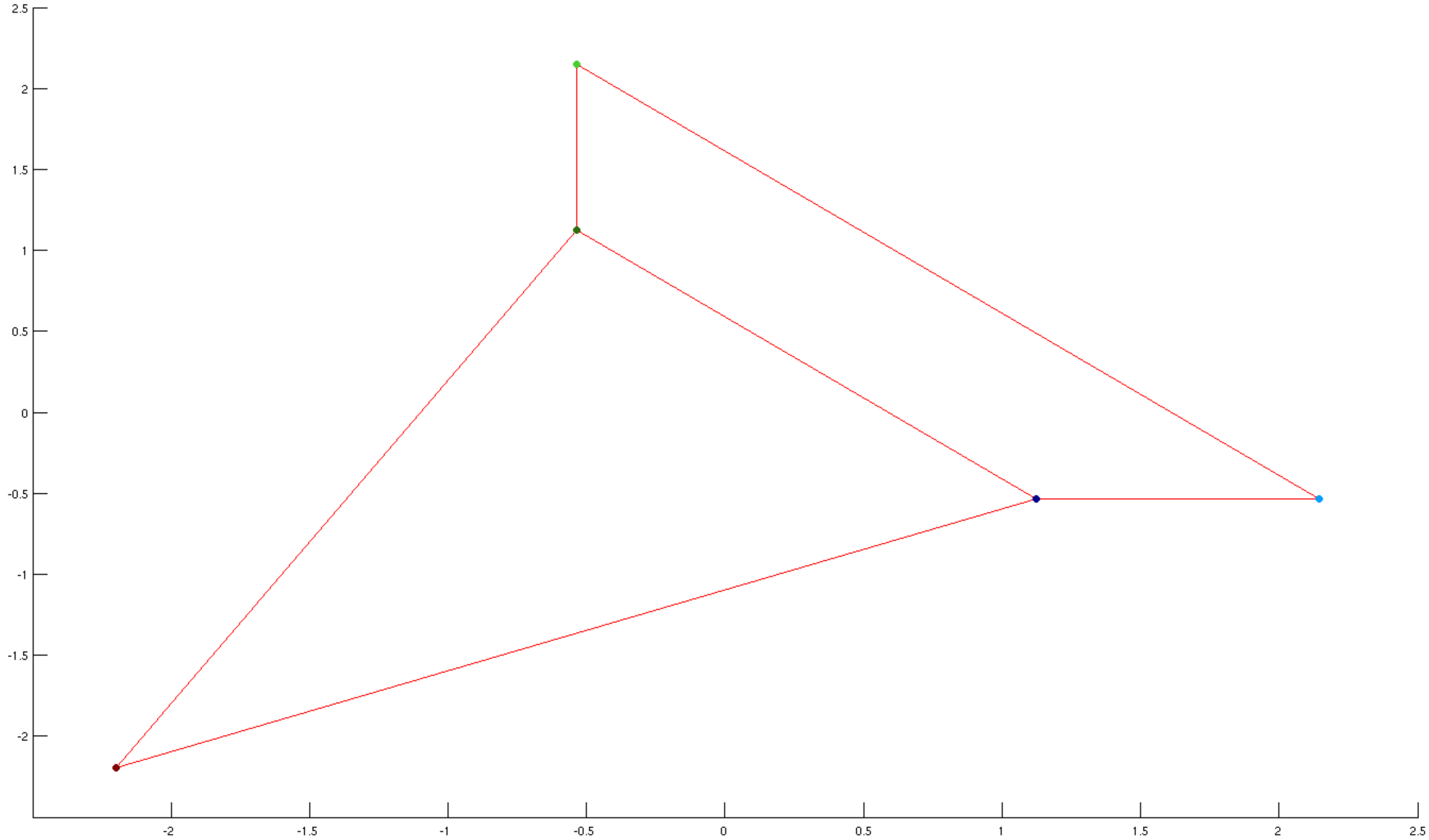

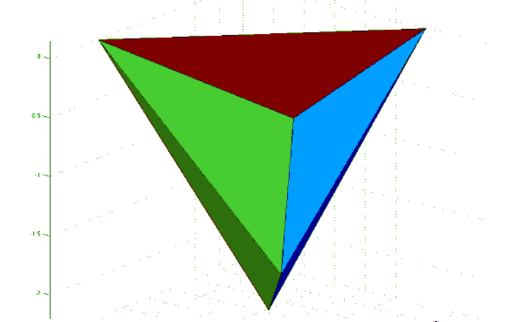

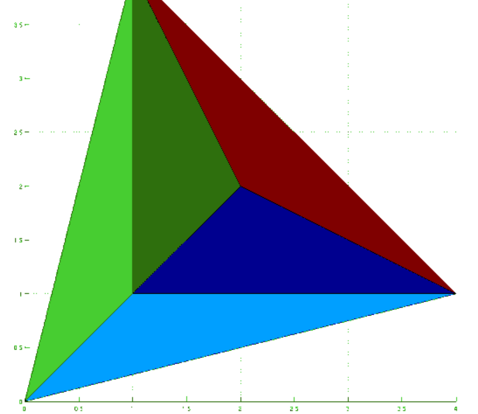

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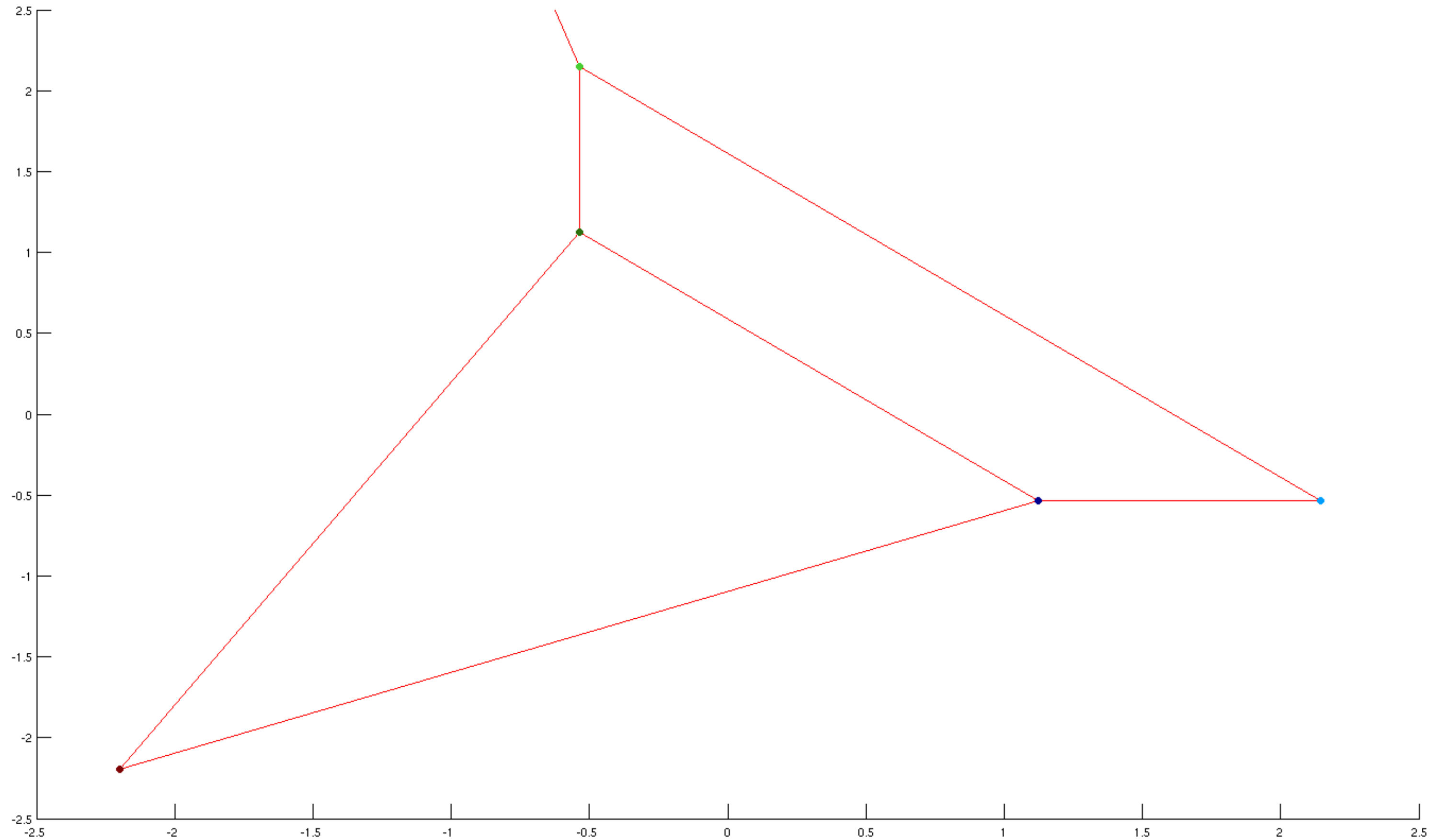

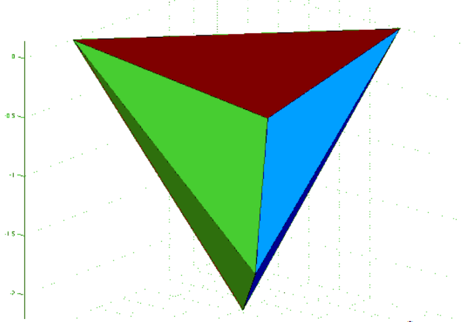

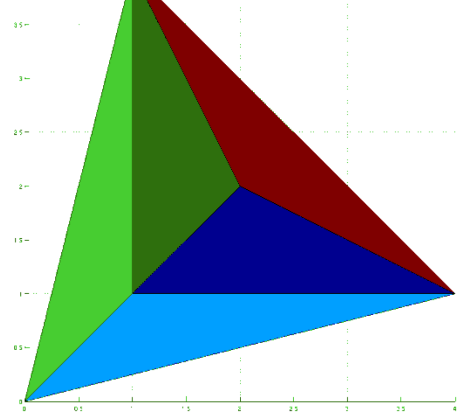

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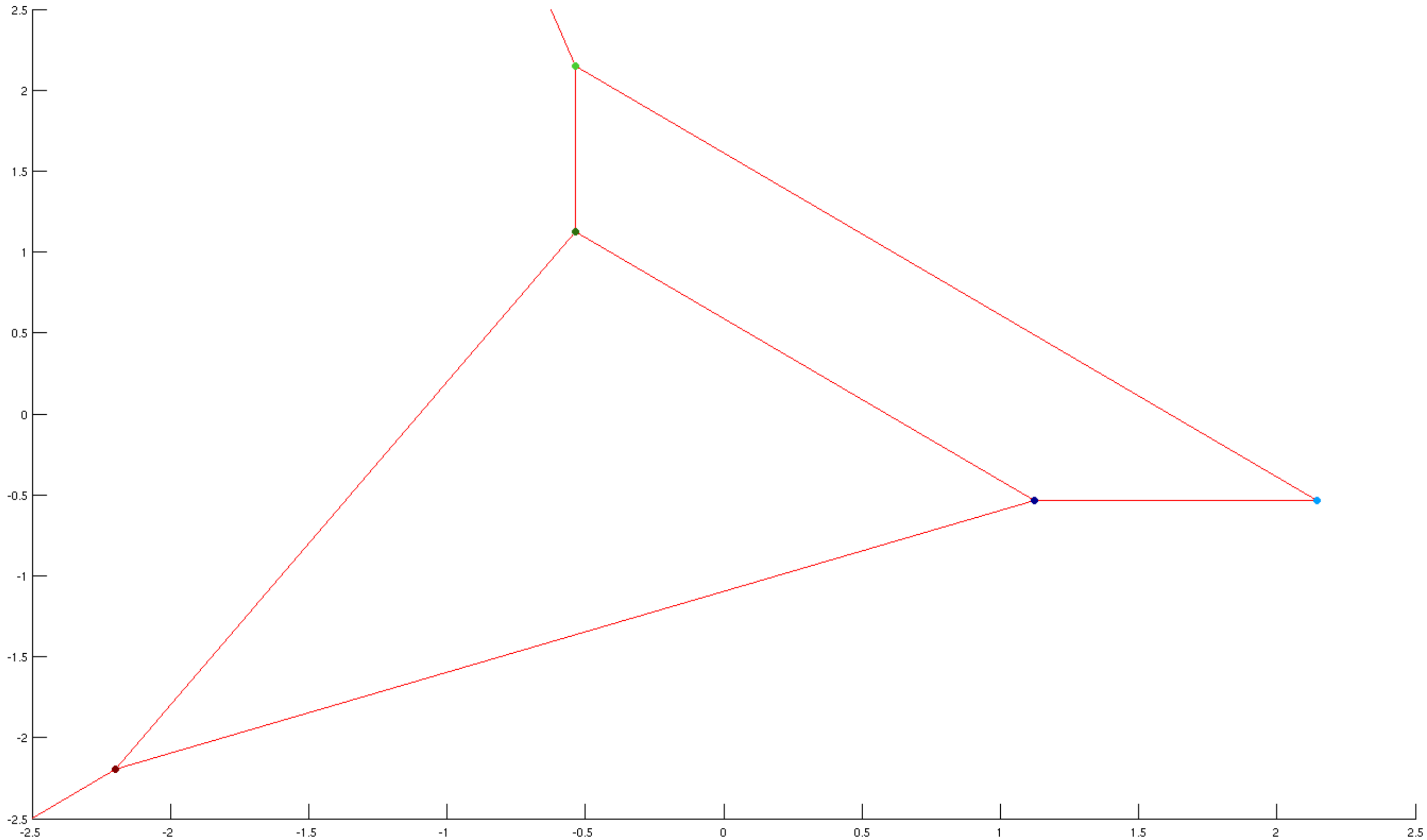

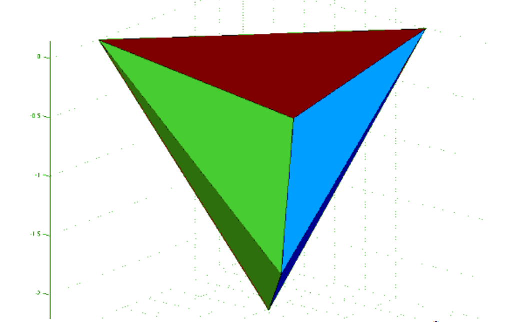

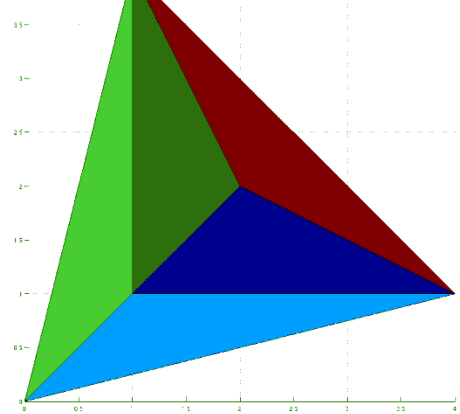

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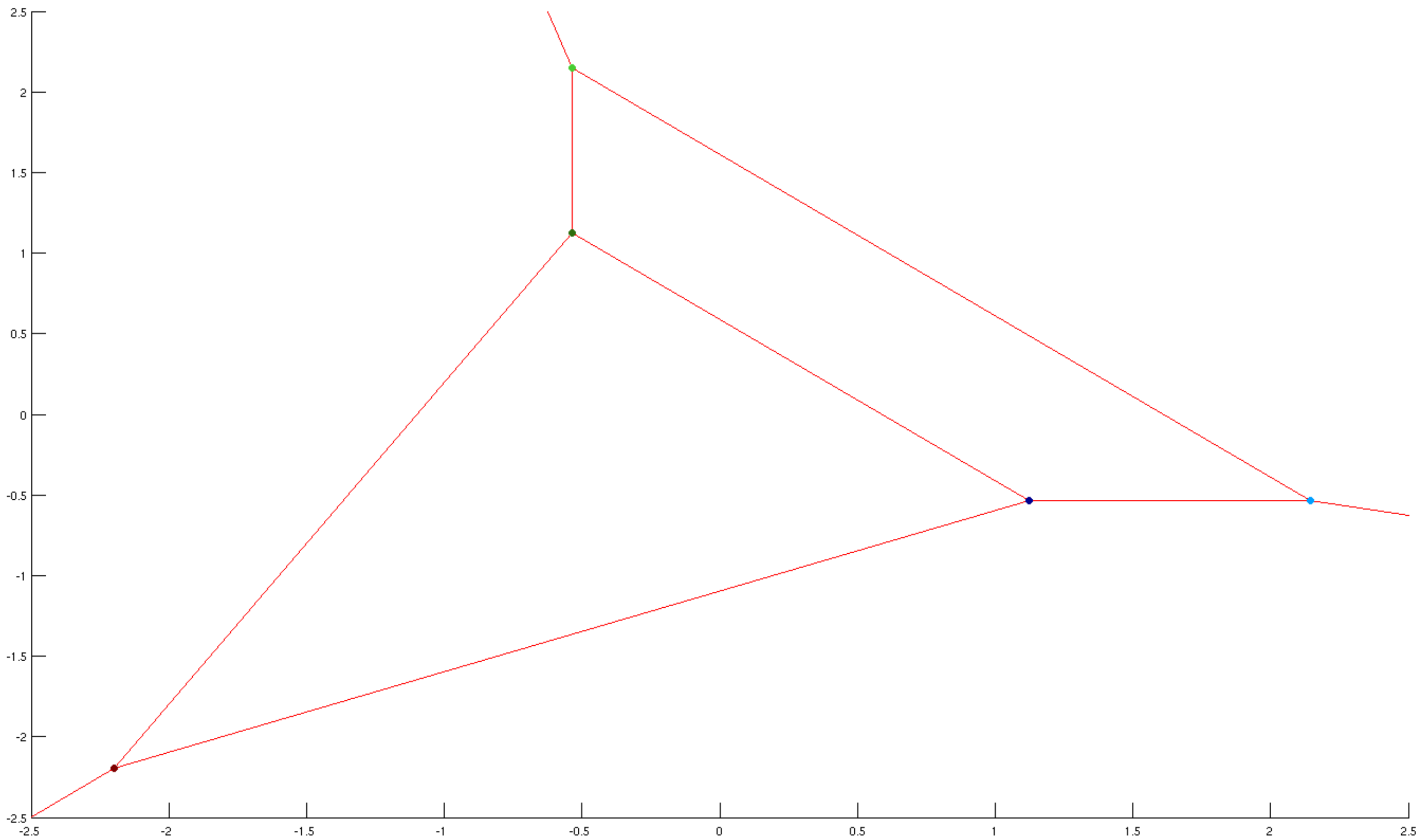

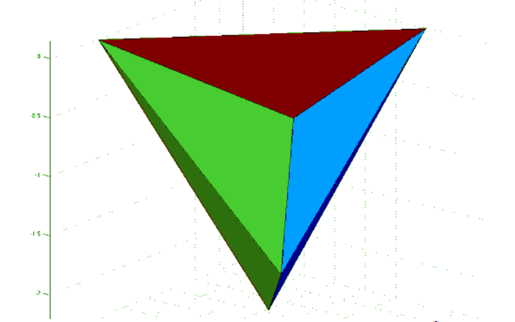

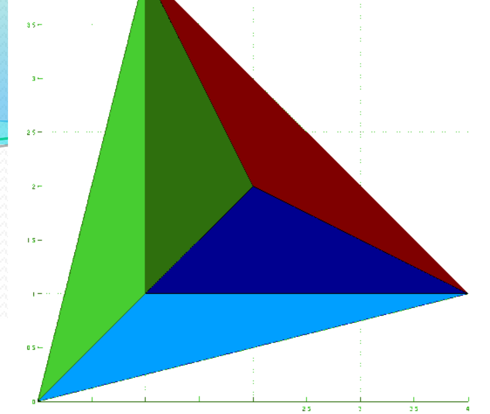

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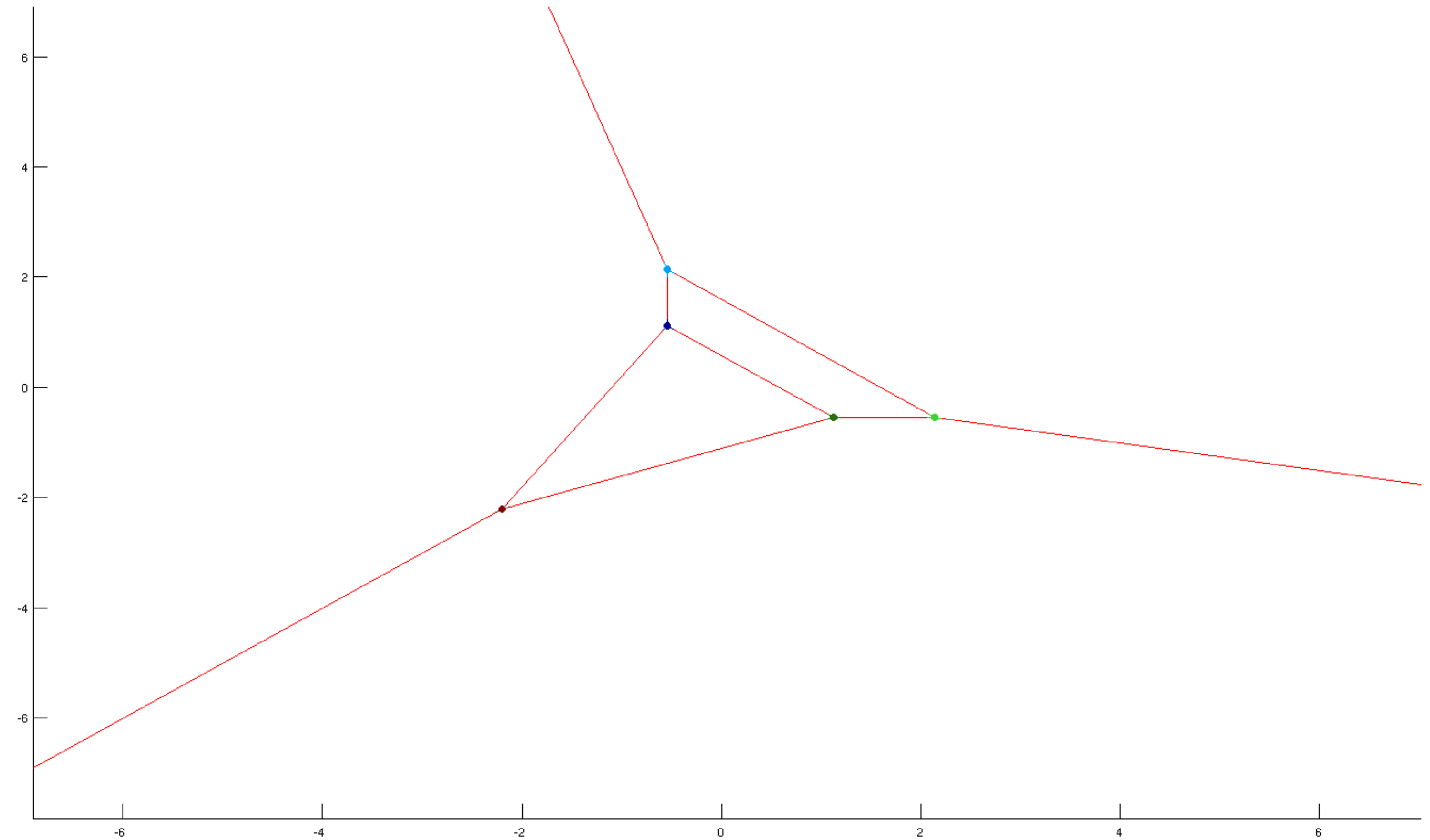

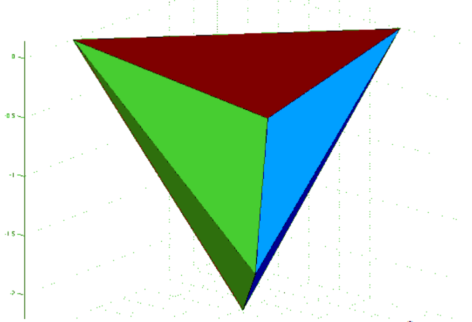

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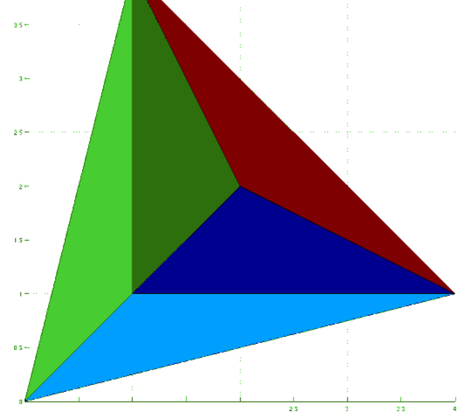

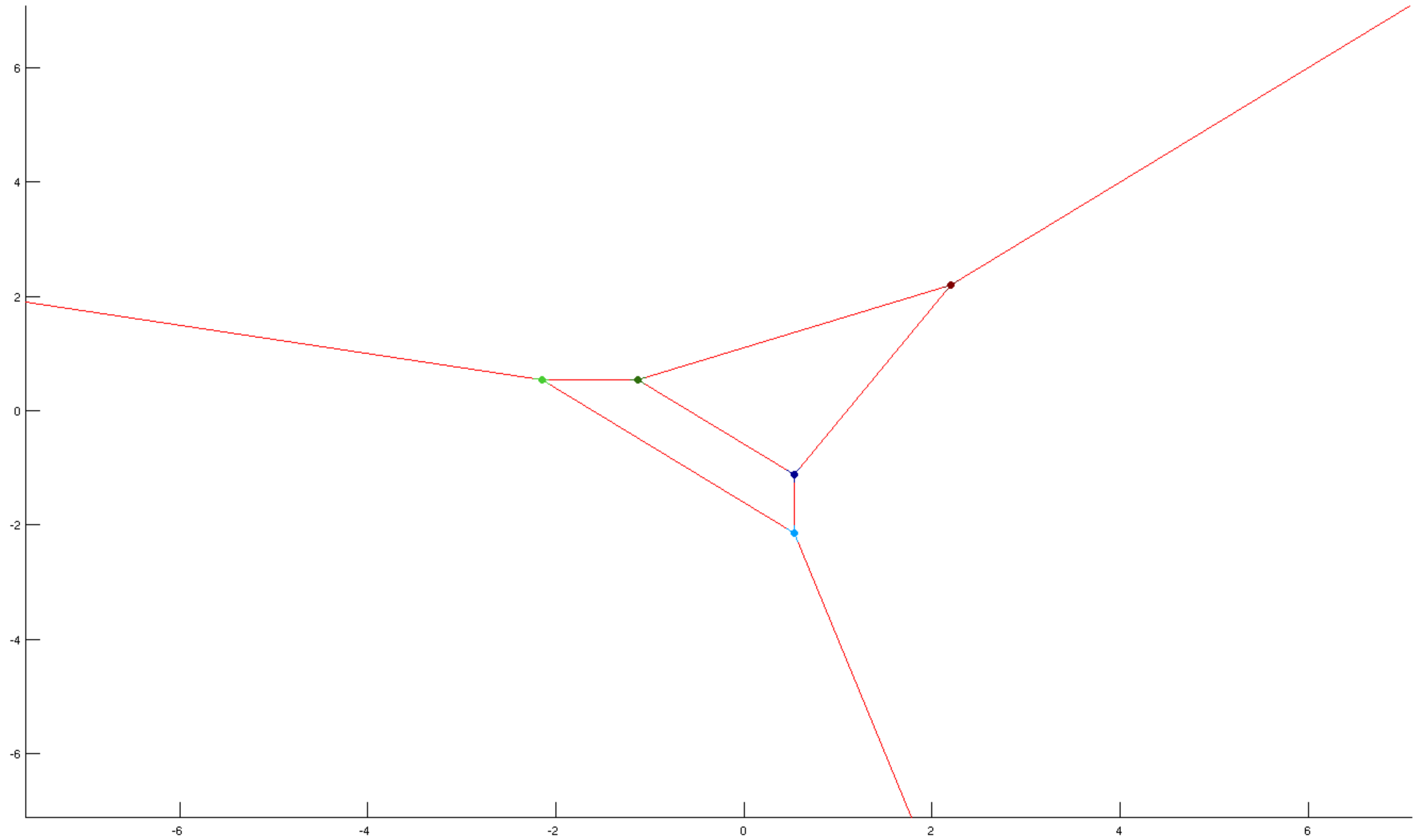

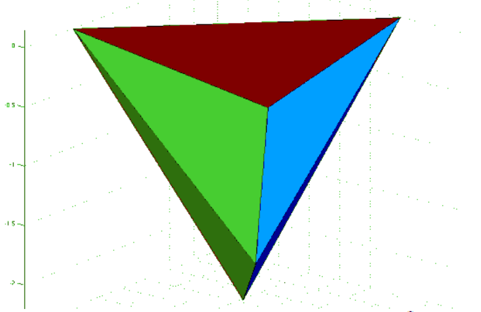

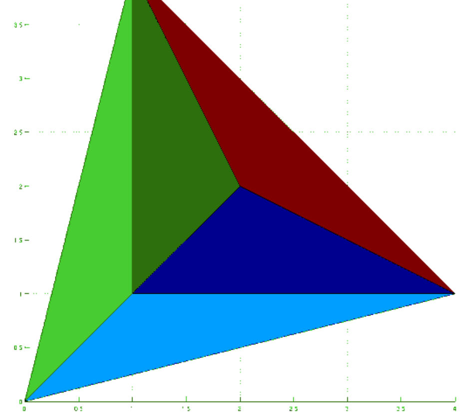

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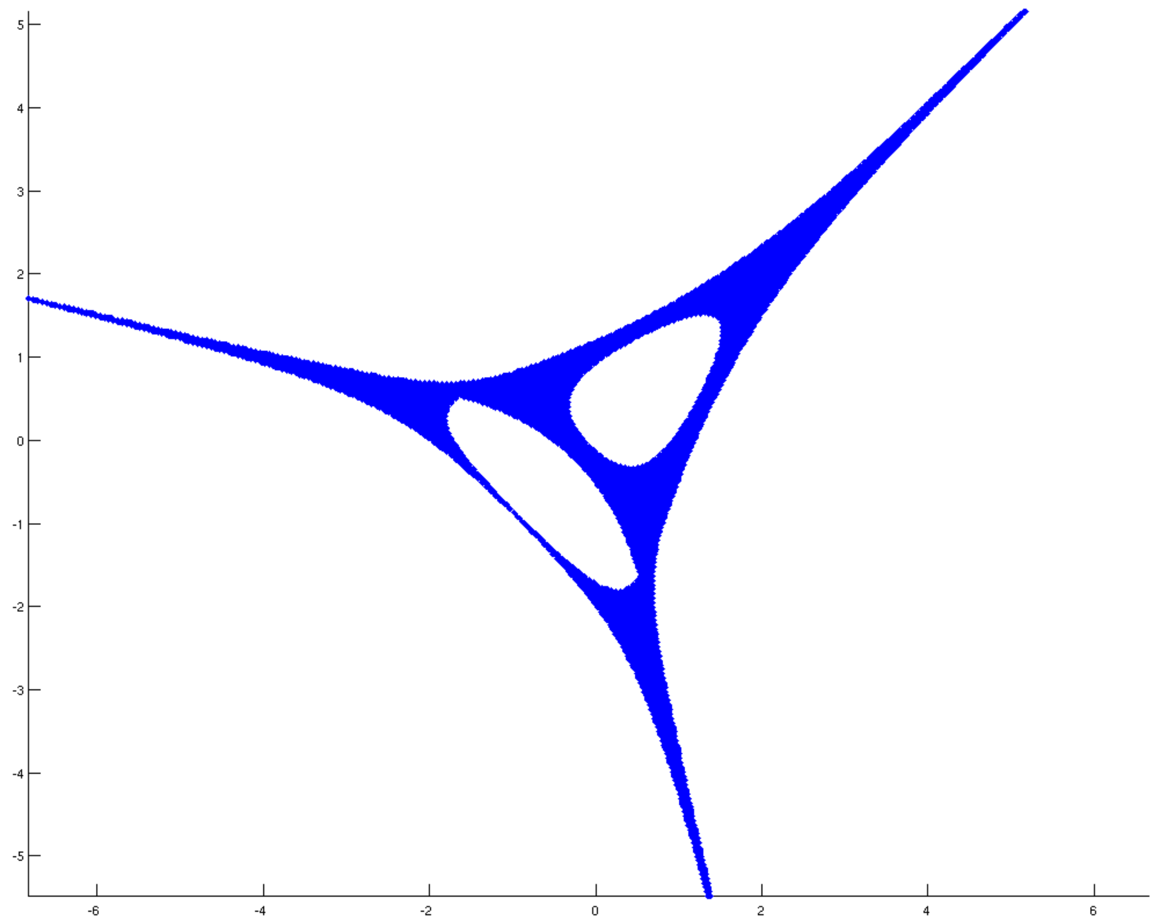
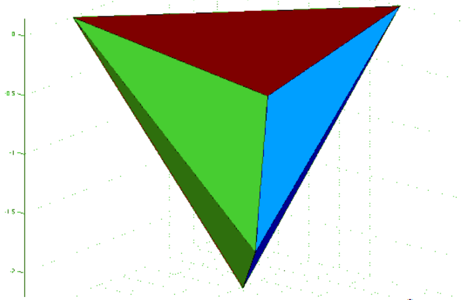
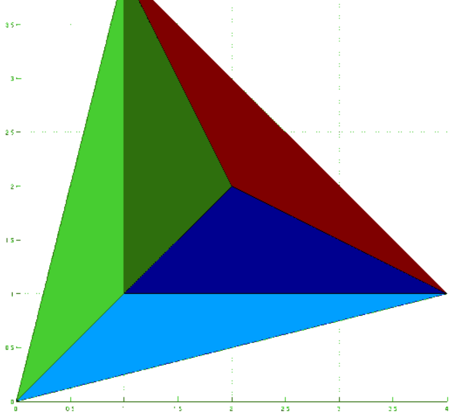
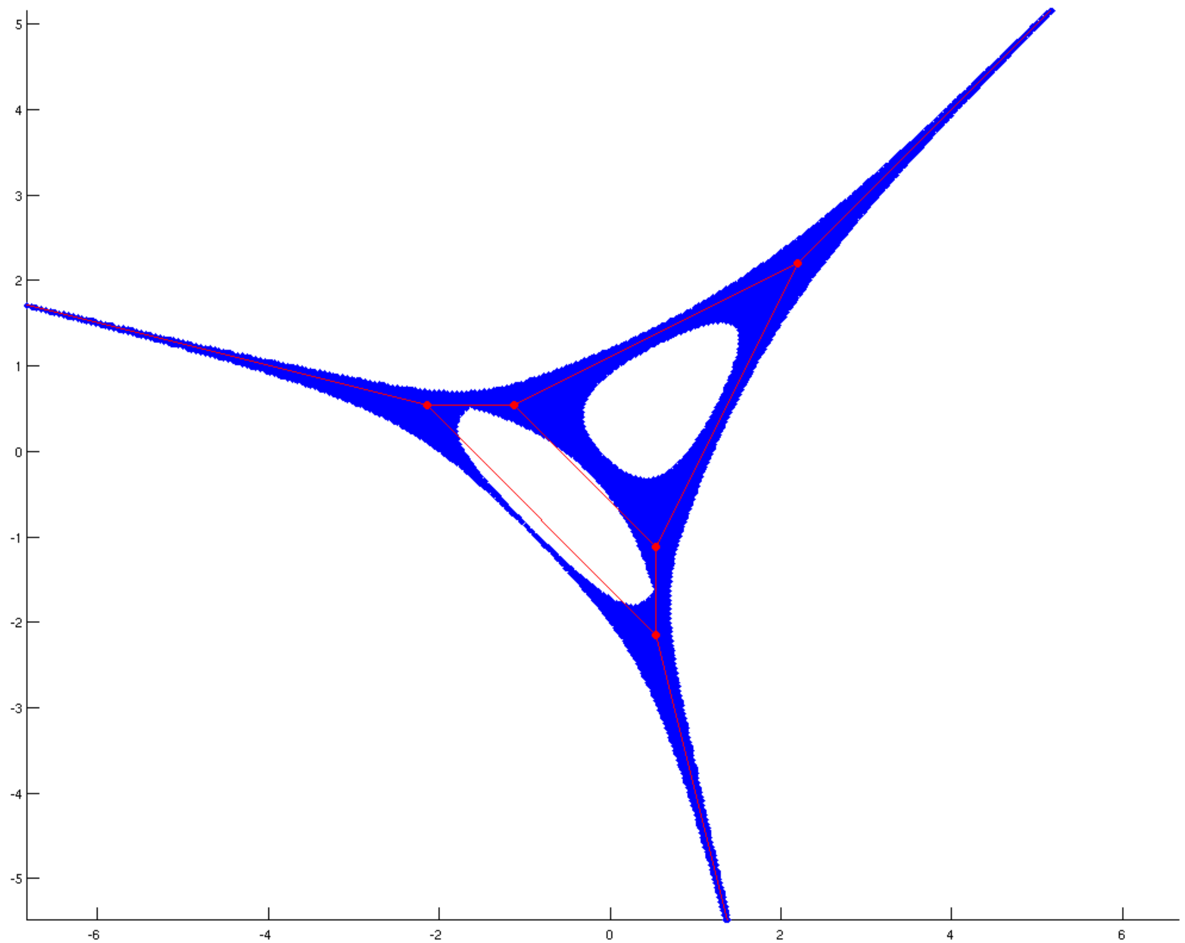


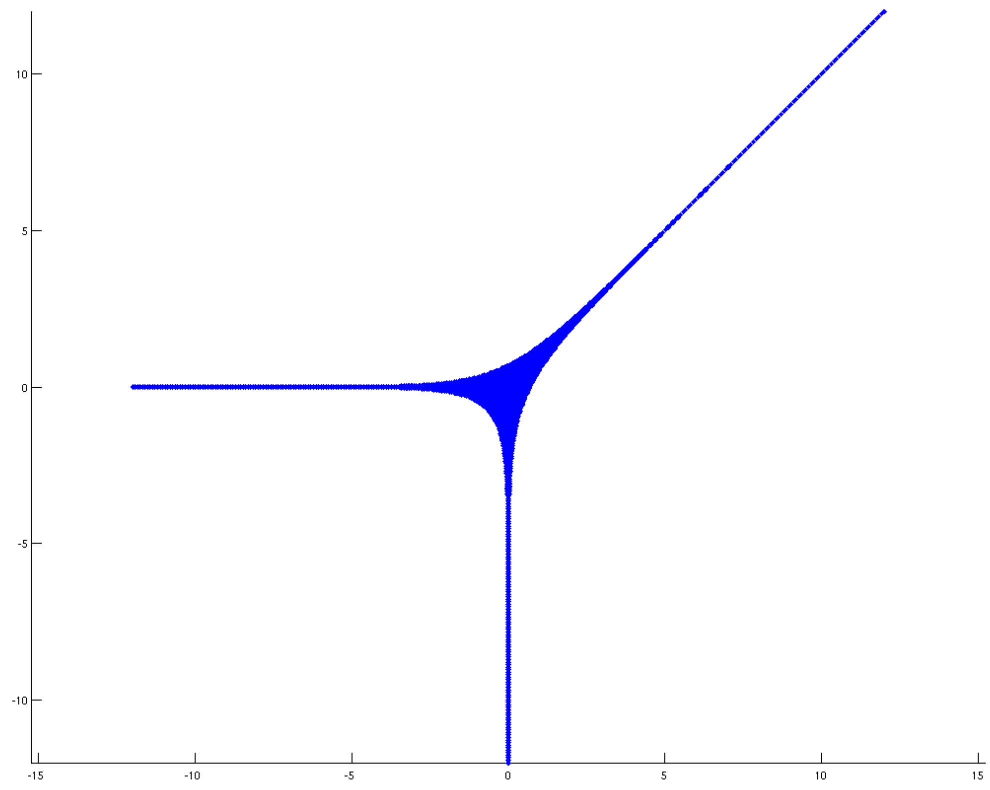


$$f(x_1, x_2) = x_1^4 x_2 + x_1 x_2^4 - 9x_1^2 x_2^2 + 5x_1 x_2 + 1$$

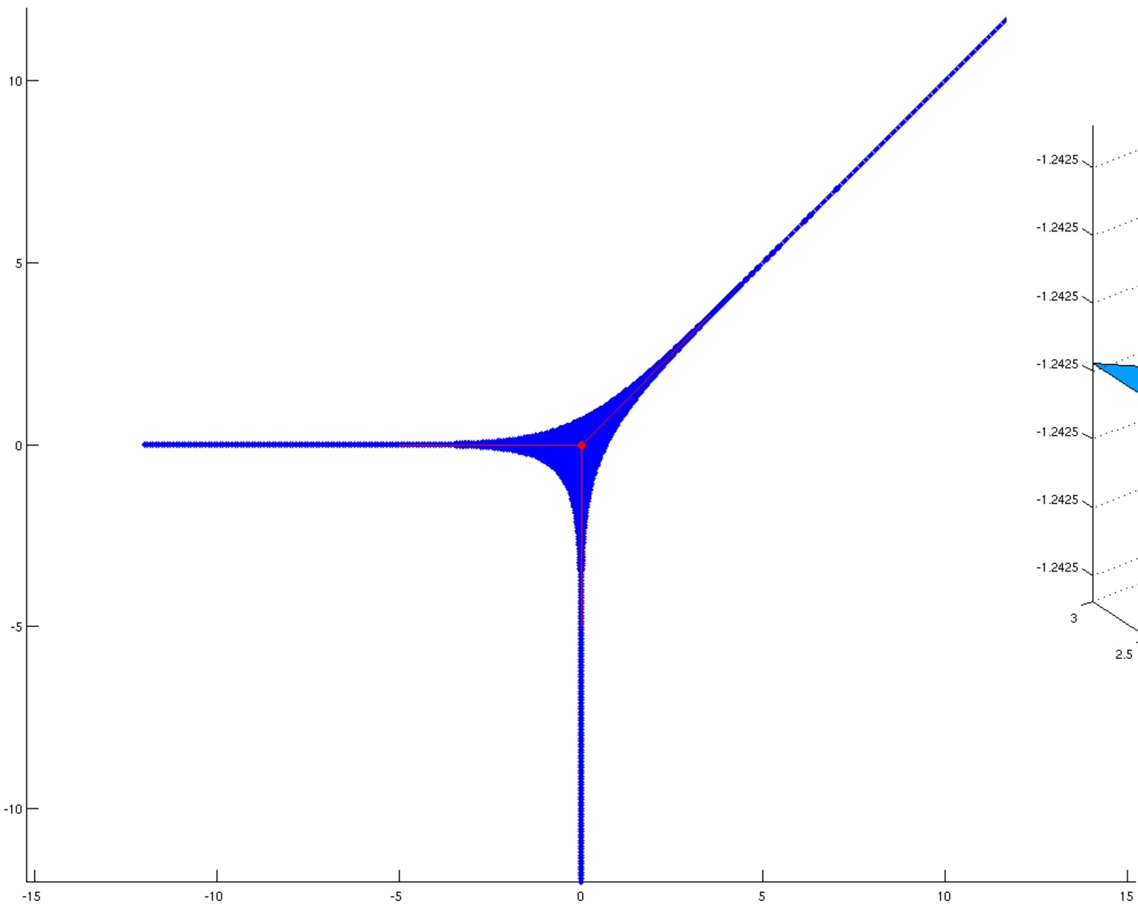


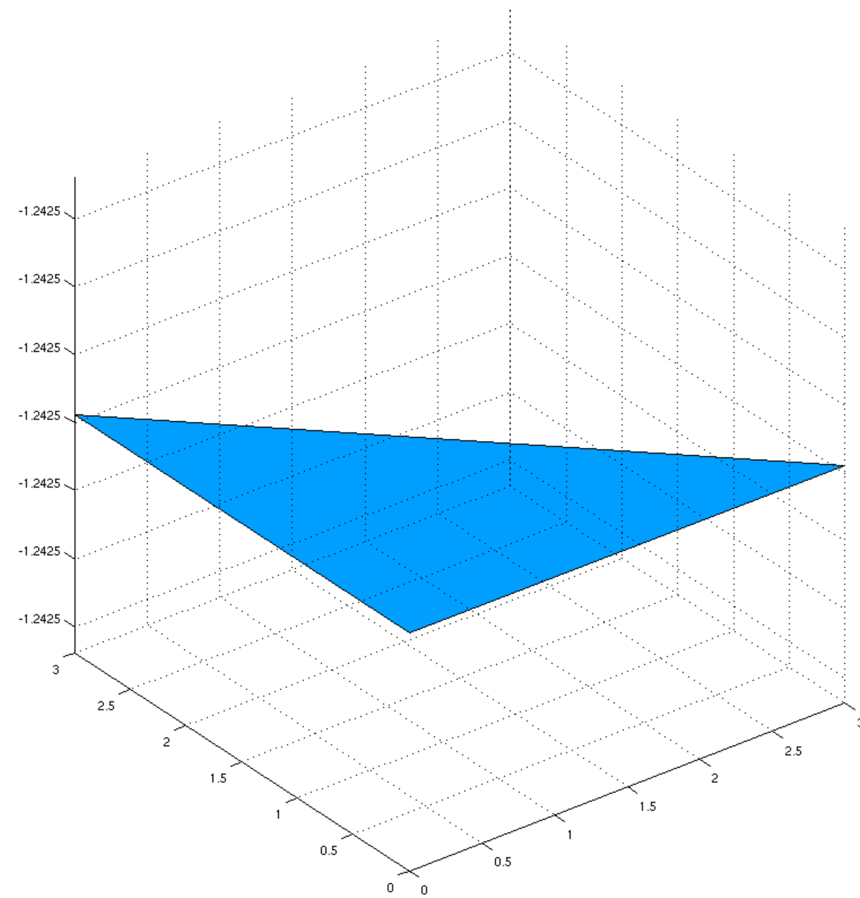


$$f(x_1, x_2) = x_1^3 + x_2^3 - 3x_1x_2 + 1$$



$$f(x_1, x_2) = x_1^3 + x_2^3 - 3x_1x_2 + 1$$





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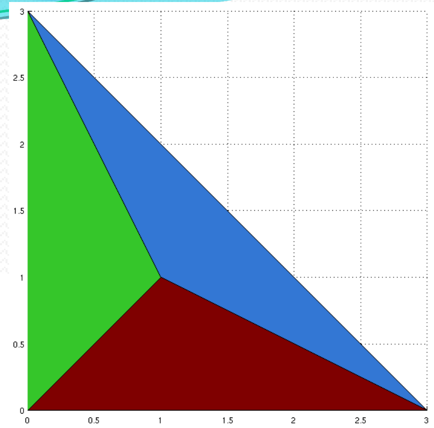
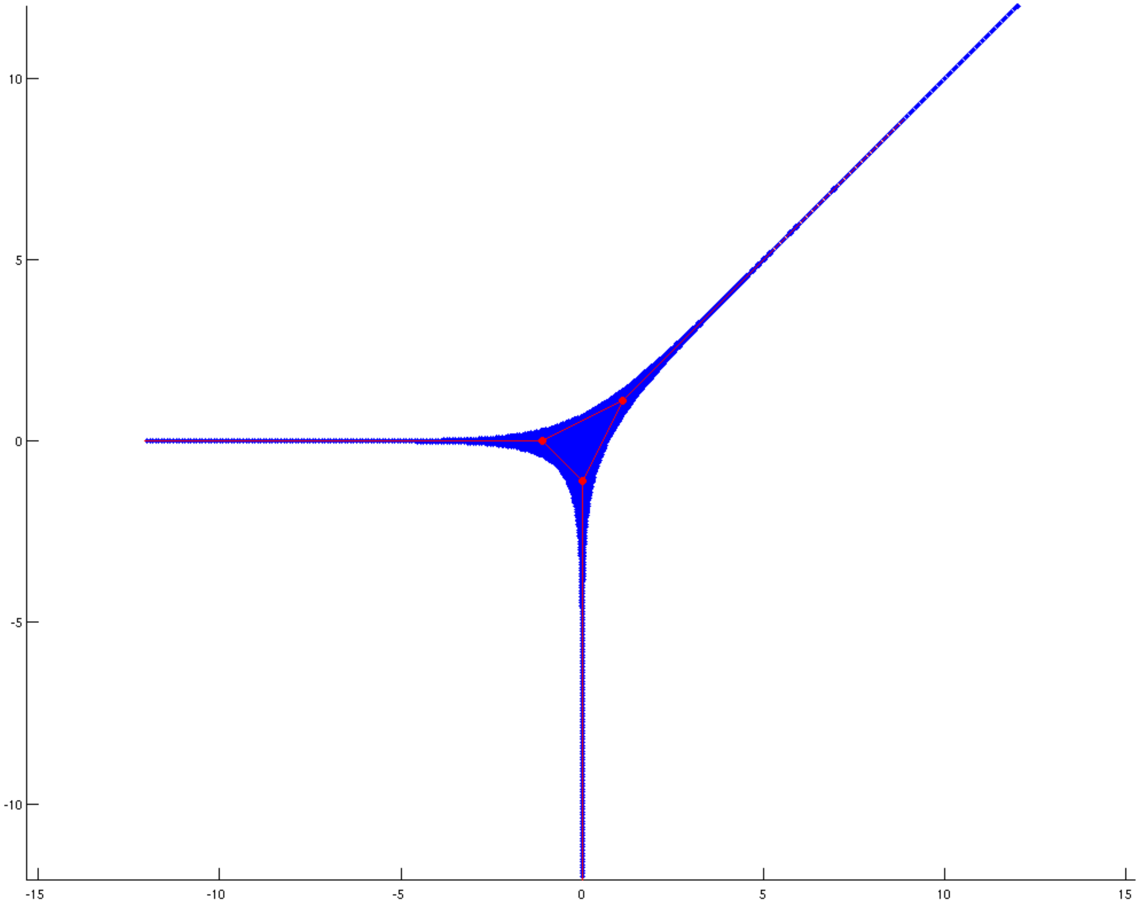
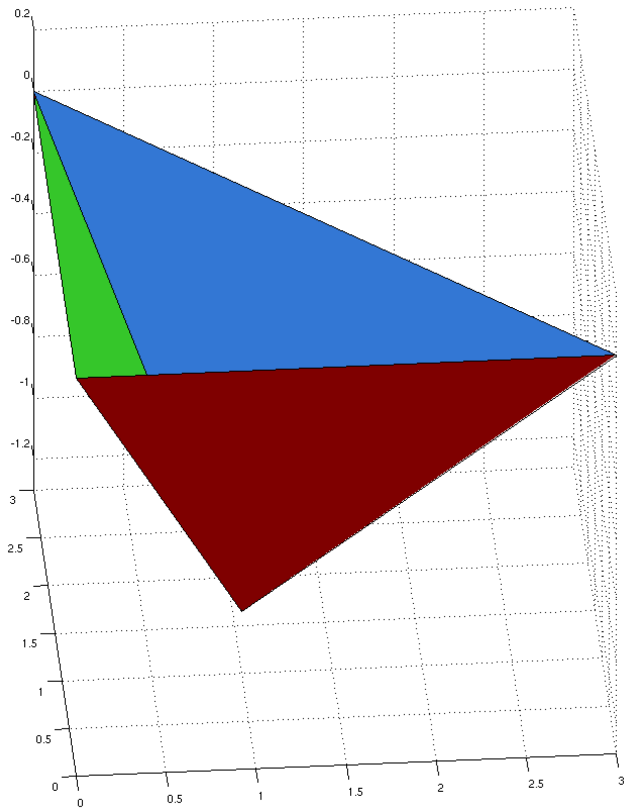
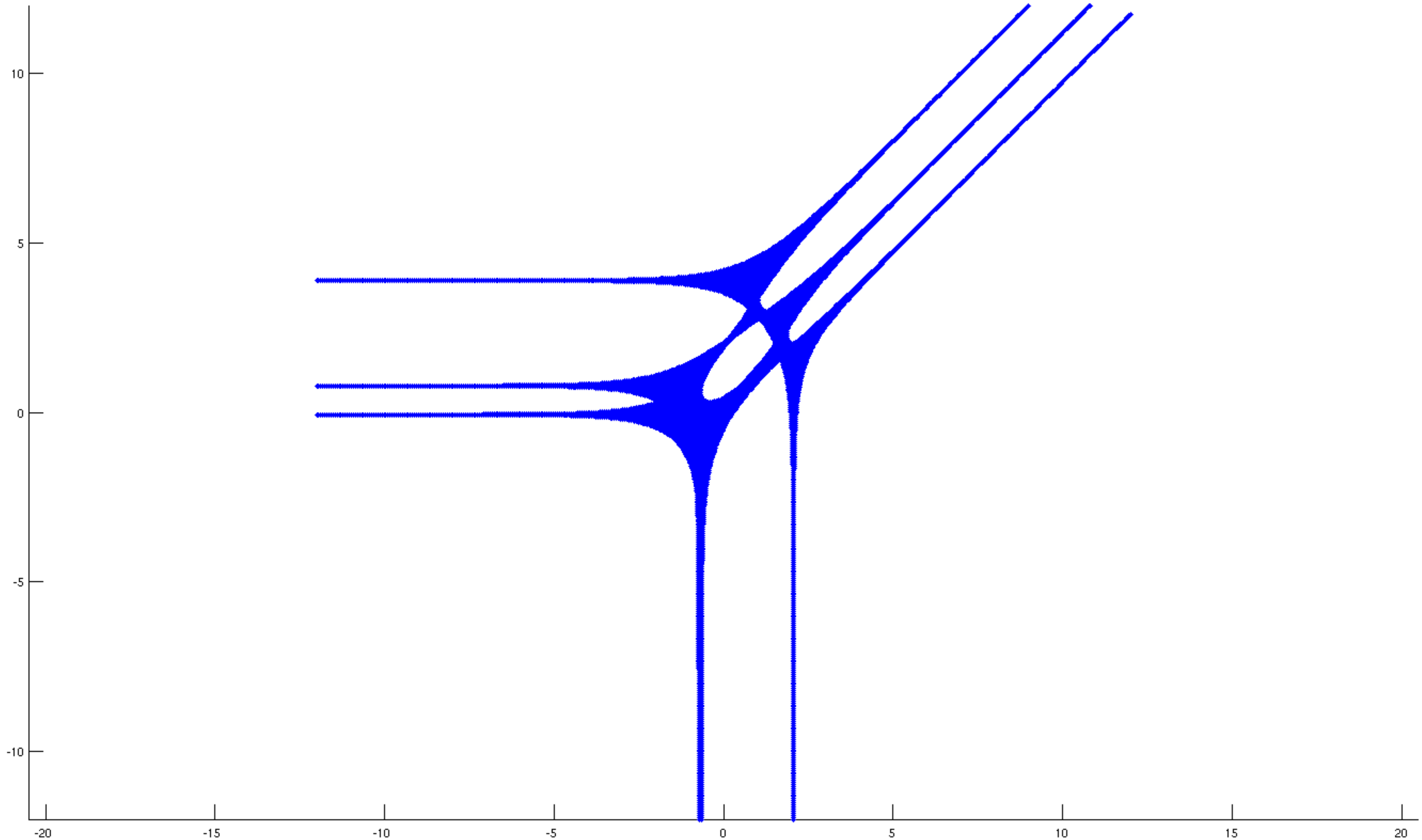


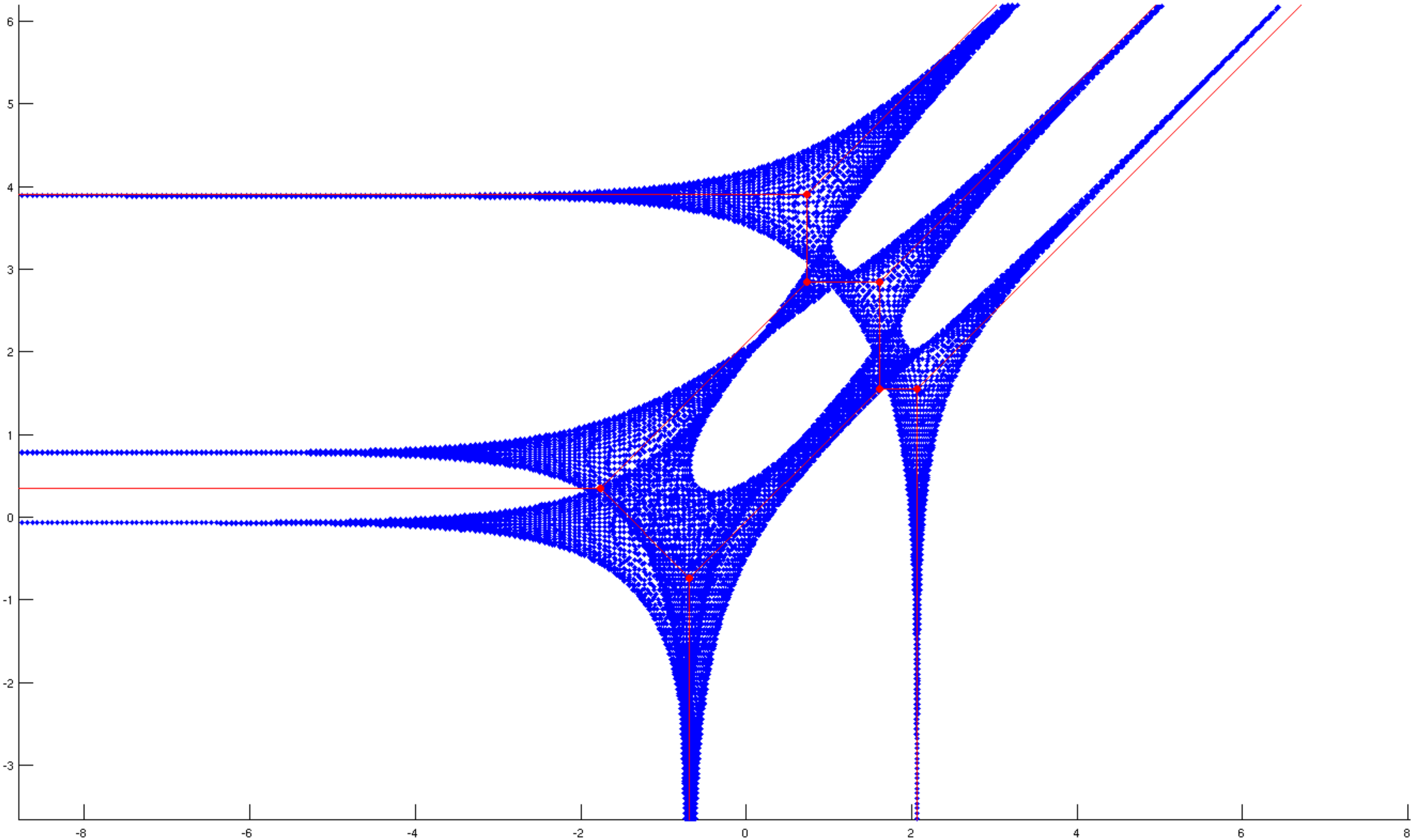




$$f(x_1, x_2) = 50x_1^3 + 83x_1^2 x_2 + 24x_1 x_2^2 + x_2^3 + 392x_1^2 + 414x_1 x_2 + 50x_2 - 28x_1 + 59x_2 - 100$$



$$f(x_1, x_2) = 50x_1^3 + 83x_1^2x_2 + 24x_1x_2^2 + x_2^3 + 392x_1^2 + 414x_1x_2 + 50x_2 - 28x_1 + 59x_2 - 100$$



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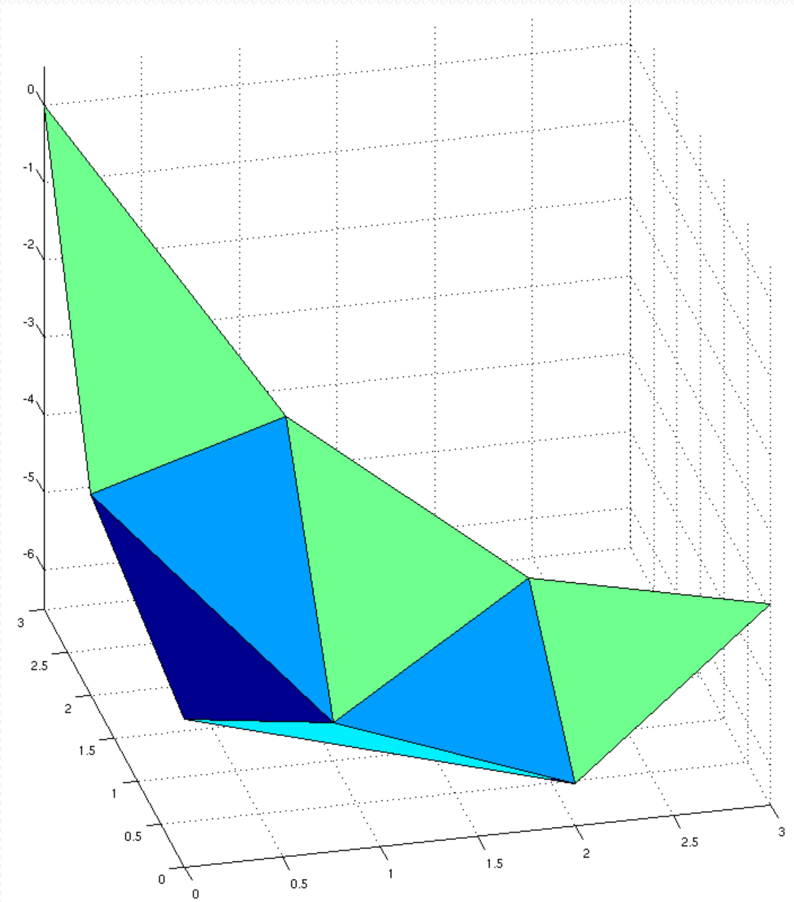
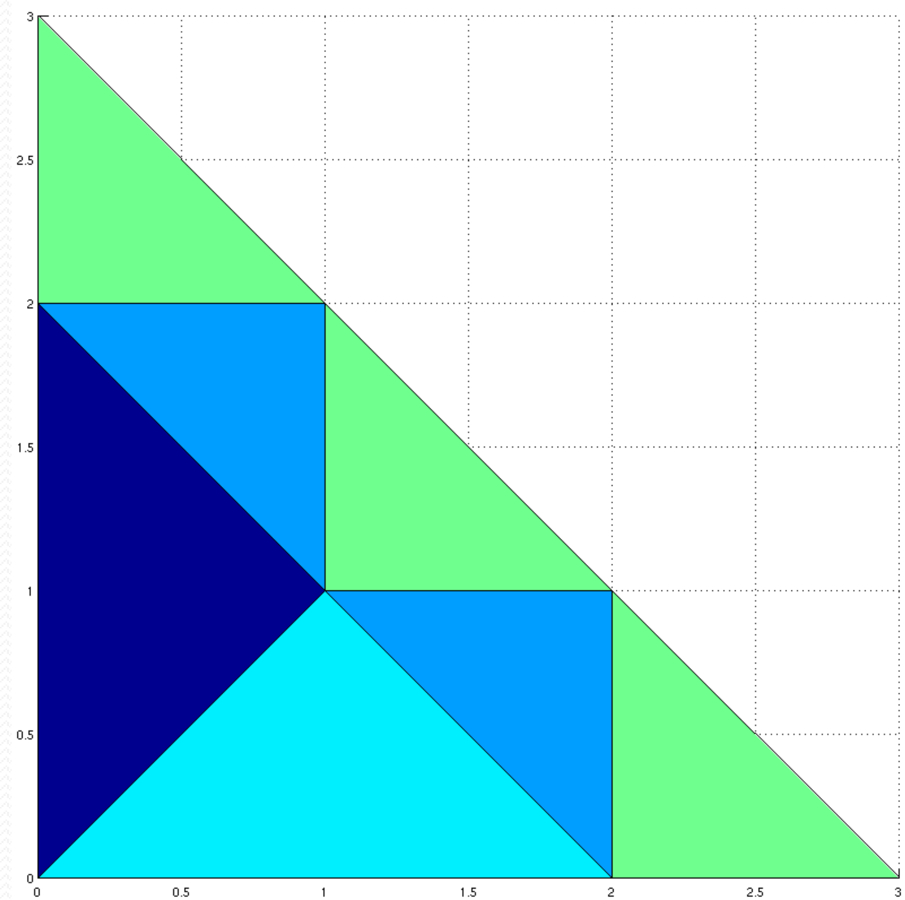
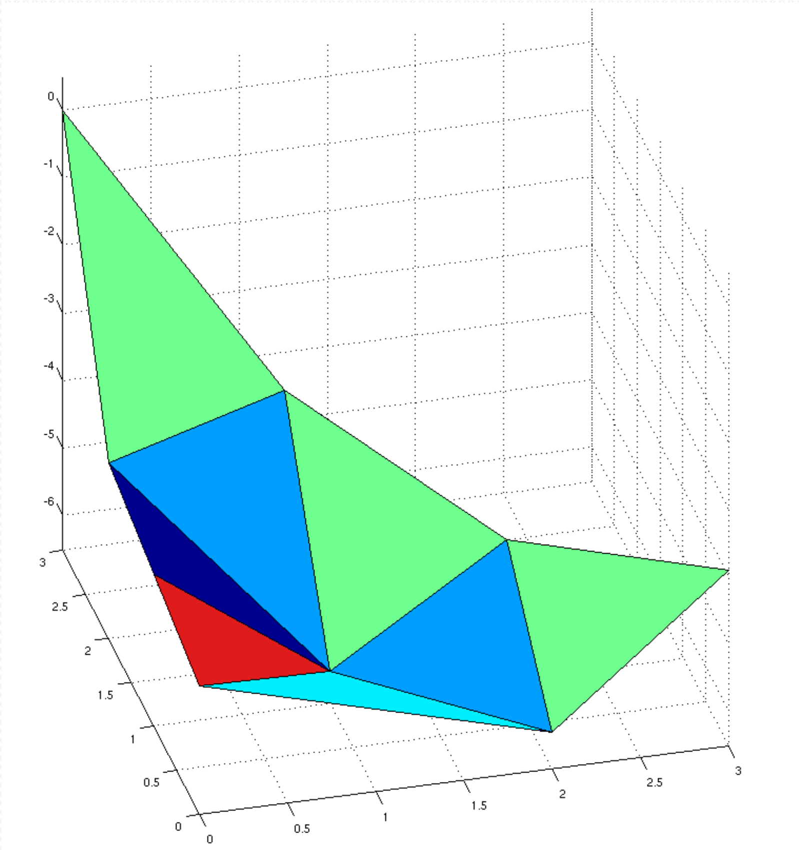
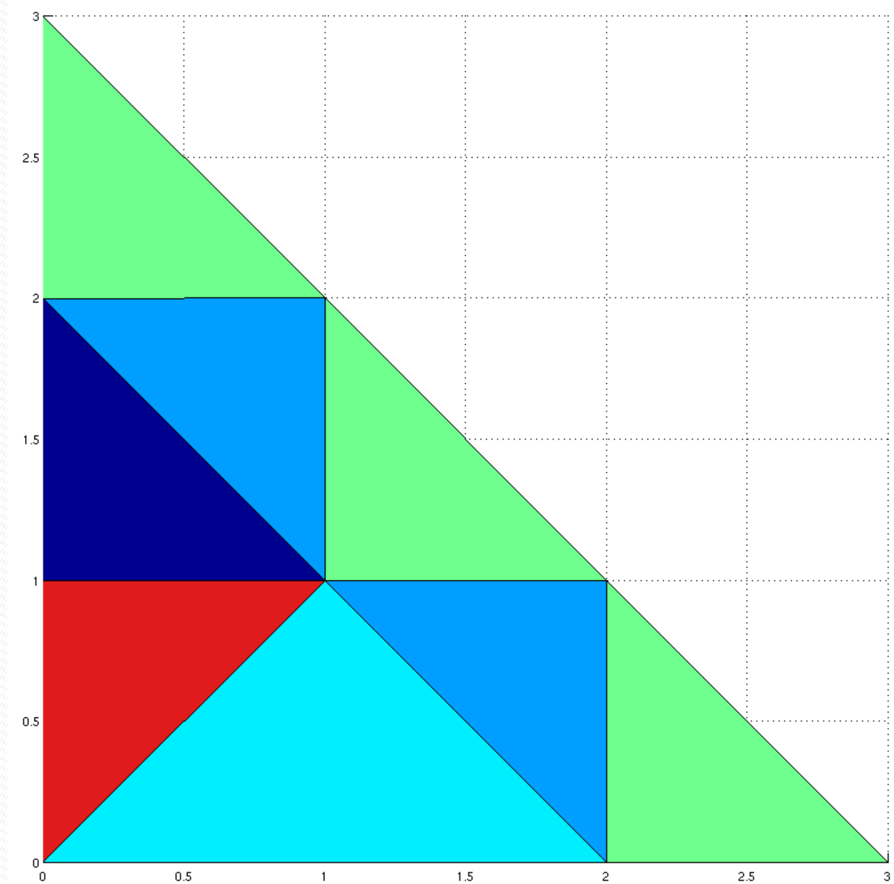


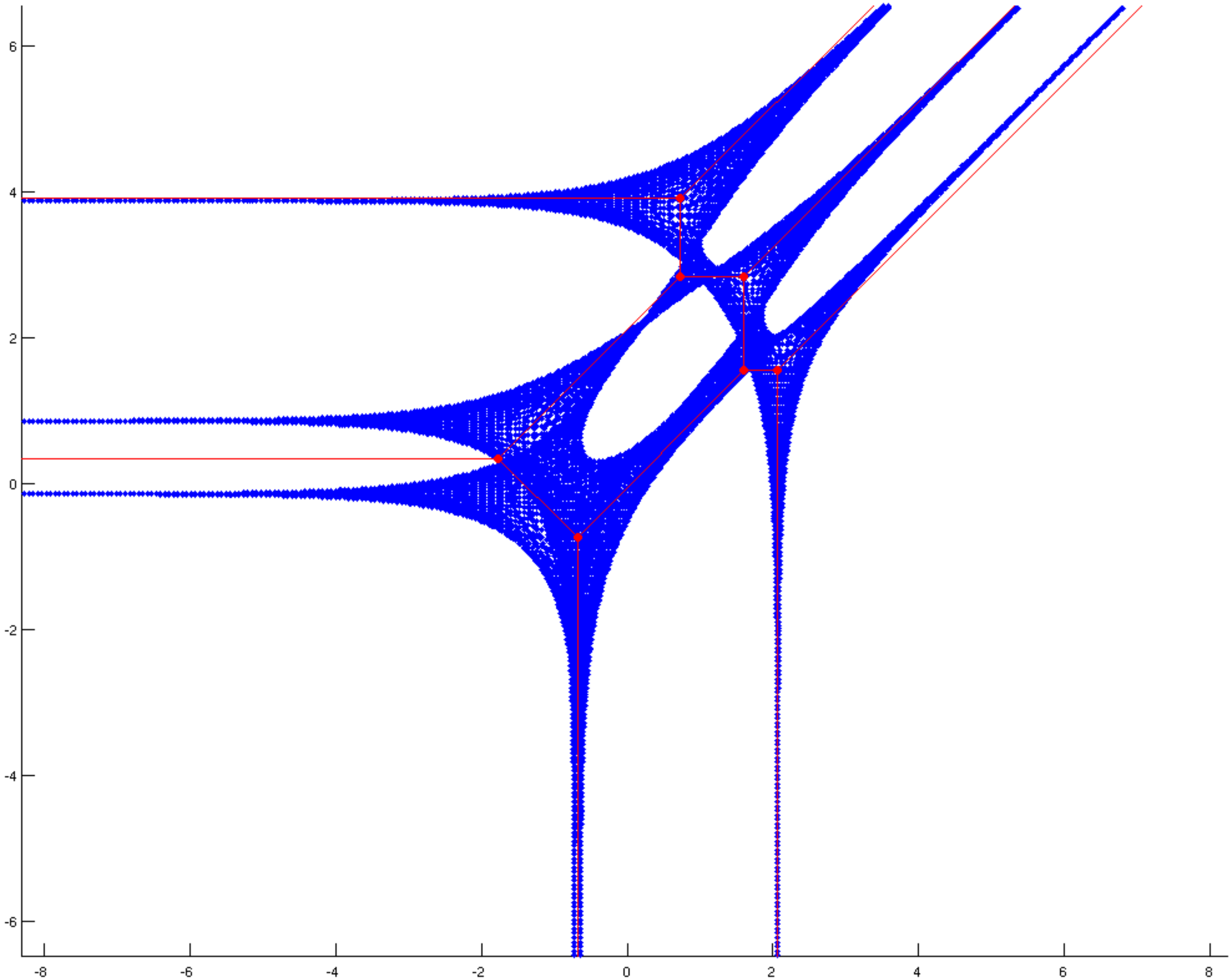


$$f(x_1, x_2) = 50x_1^3 + 83x_1^2 x_2 + 24x_1 x_2^2 + x_2^3 + 392x_1^2 + 414x_1 x_2 + 50x_2 - 28x_1 + 50\sqrt{2}x_2 - 100$$

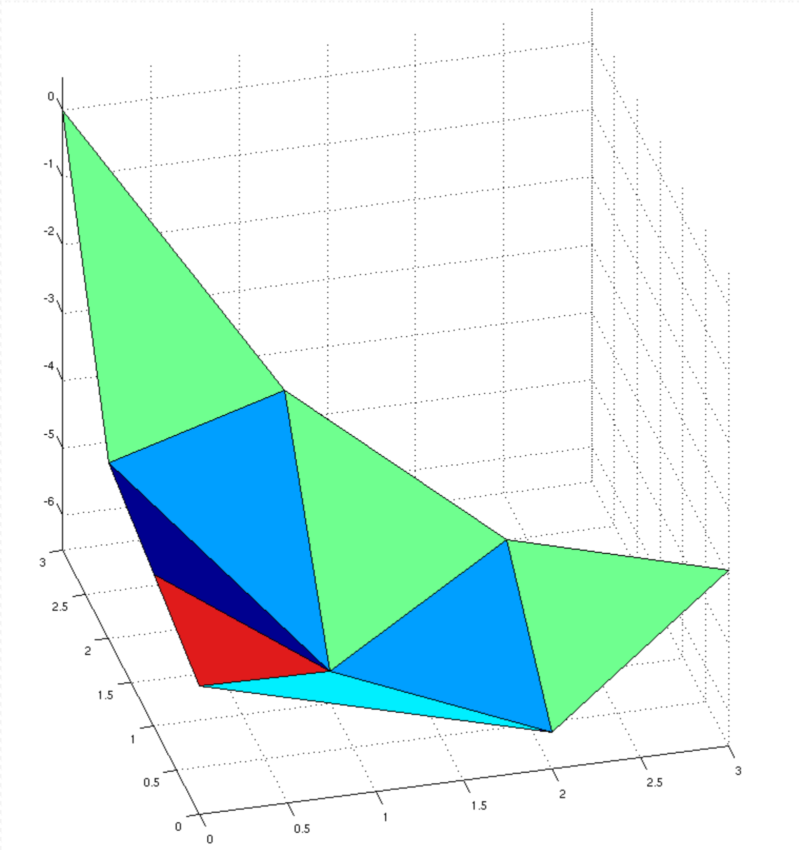


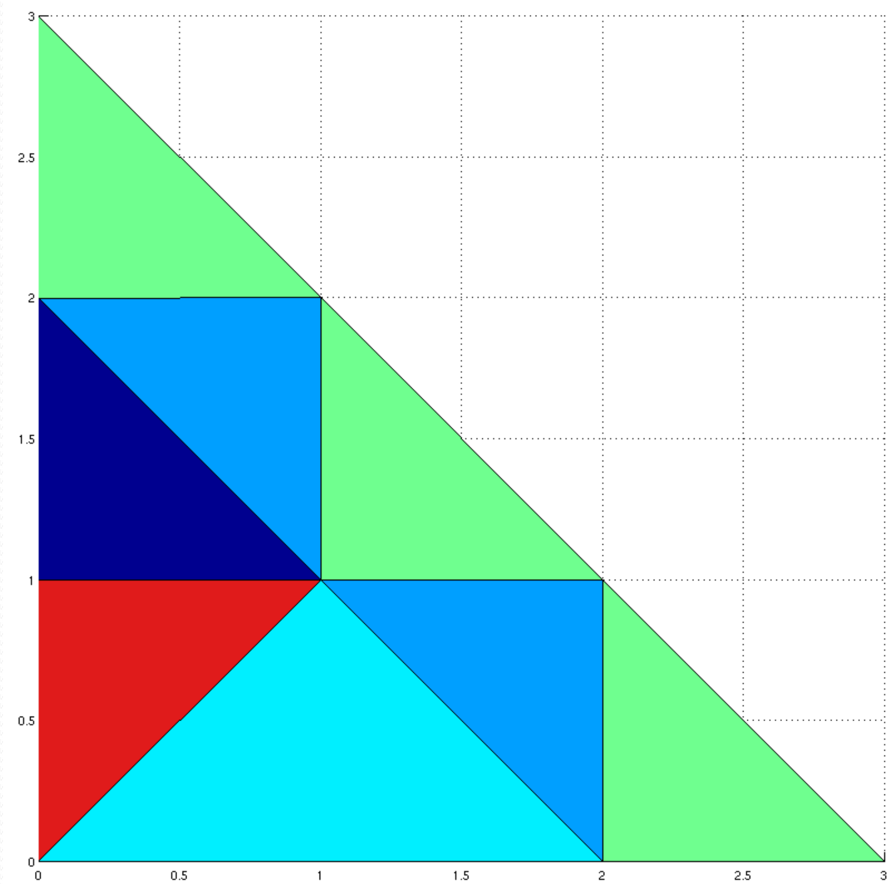


$$f(x_1,x_2) = 50x_1^3 + 83x_1^2x_2 + 24x_1x_2^2 + x_2^3 + 392x_1^2 + 414x_1x_2 + 50x_2 - 28x_1 + 50\sqrt{2}x_2 - 100$$

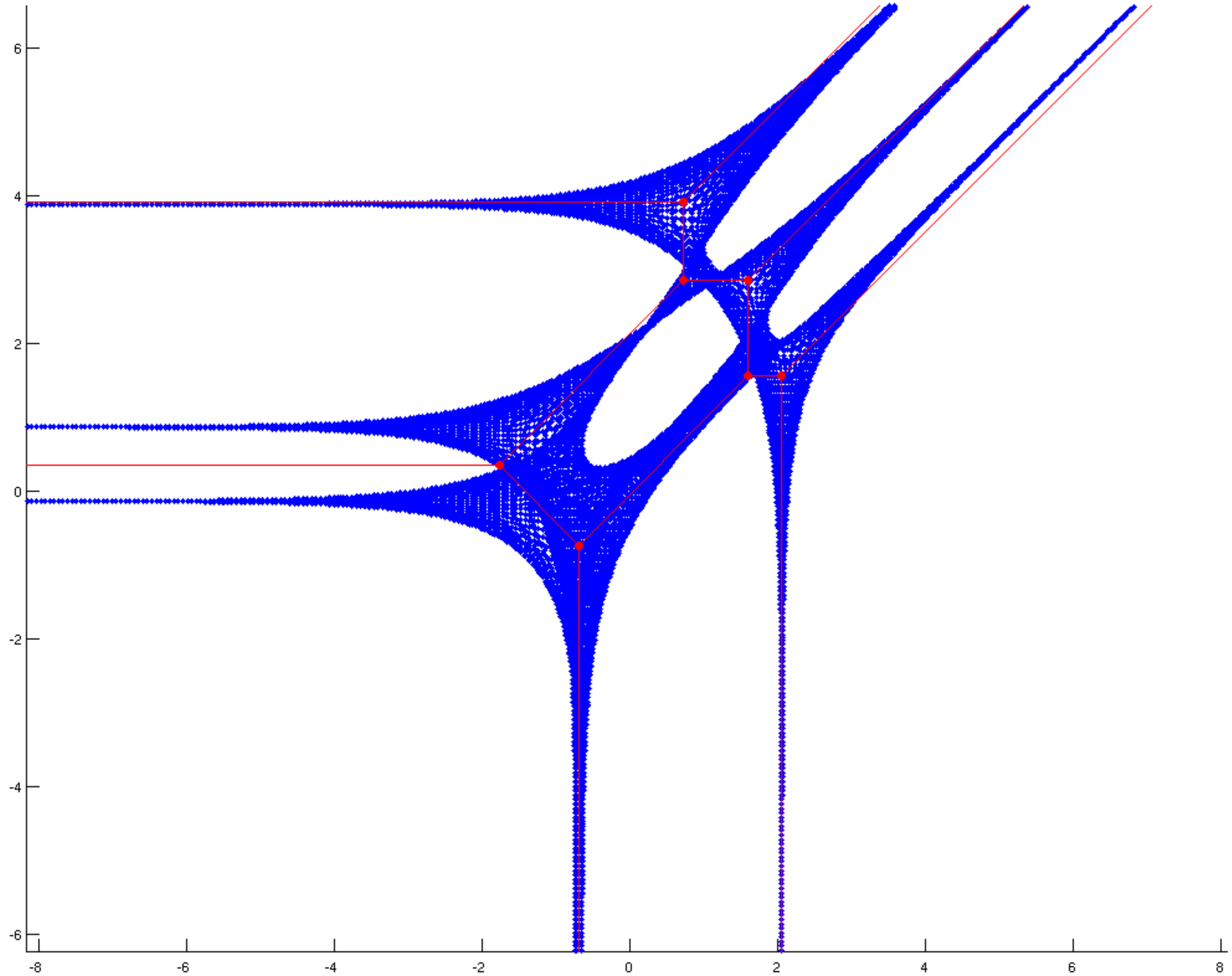


$$f(x_1, x_2) = 50x_1^3 + 83x_1^2x_2 + 24x_1x_2^2 + x_2^3 + 392x_1^2 + 414x_1x_2 + 50x_2 - 28x_1 + (50\sqrt{2} + .1)x_2 - 100$$

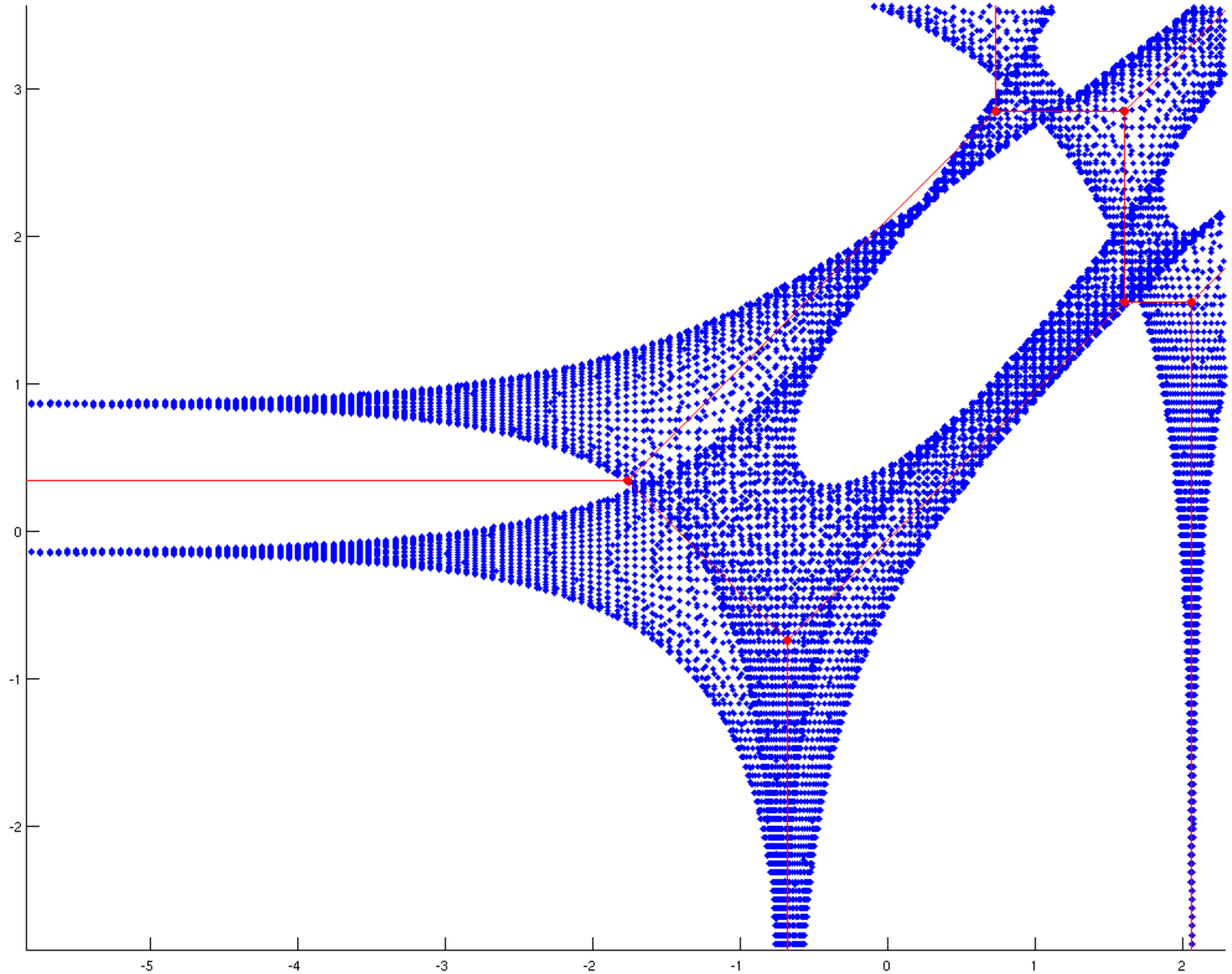




$$f(x_1, x_2) = 50x_1^3 + 83x_1^2x_2 + 24x_1x_2^2 + x_2^3 + 392x_1^2 + 414x_1x_2 + 50x_2 - 28x_1 + (50\sqrt{2} + .1)x_2 - 100$$



$$f(x_1, x_2) = 50x_1^3 + 83x_1^2 x_2 + 24x_1 x_2^2 + x_2^3 + 392x_1^2 + 414x_1 x_2 + 50x_2 - 28x_1 + (50\sqrt{2} + .1)x_2 - 100$$



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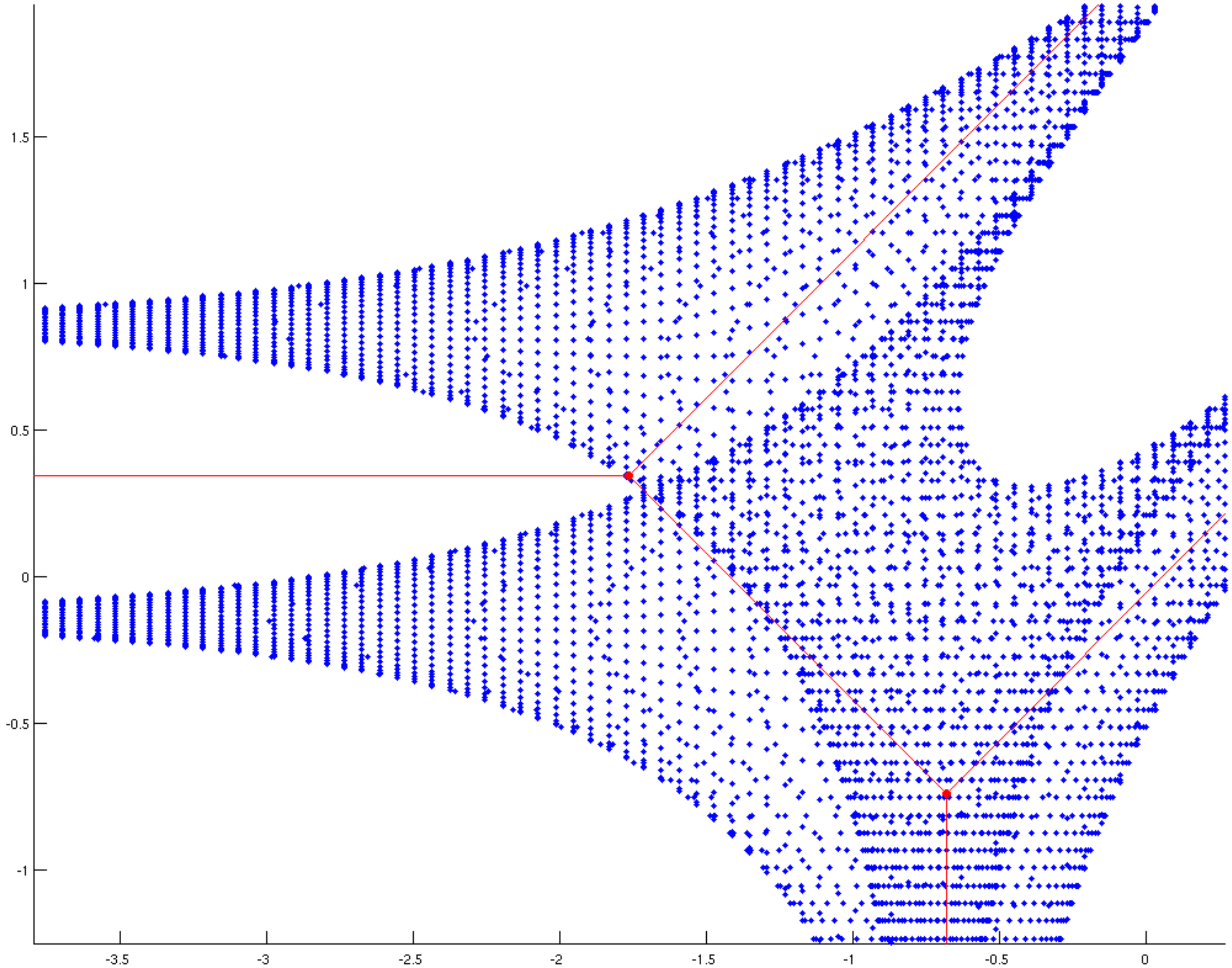
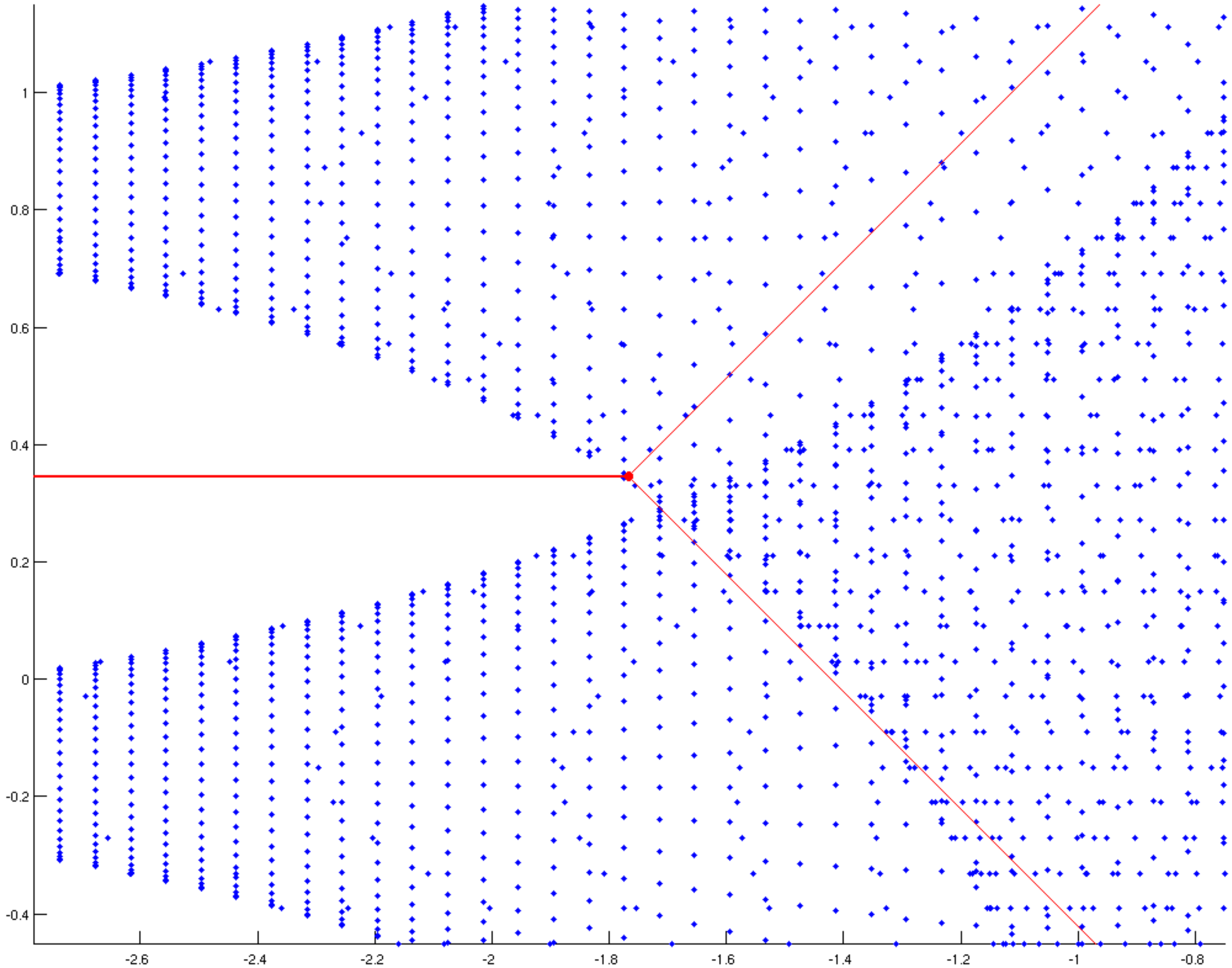
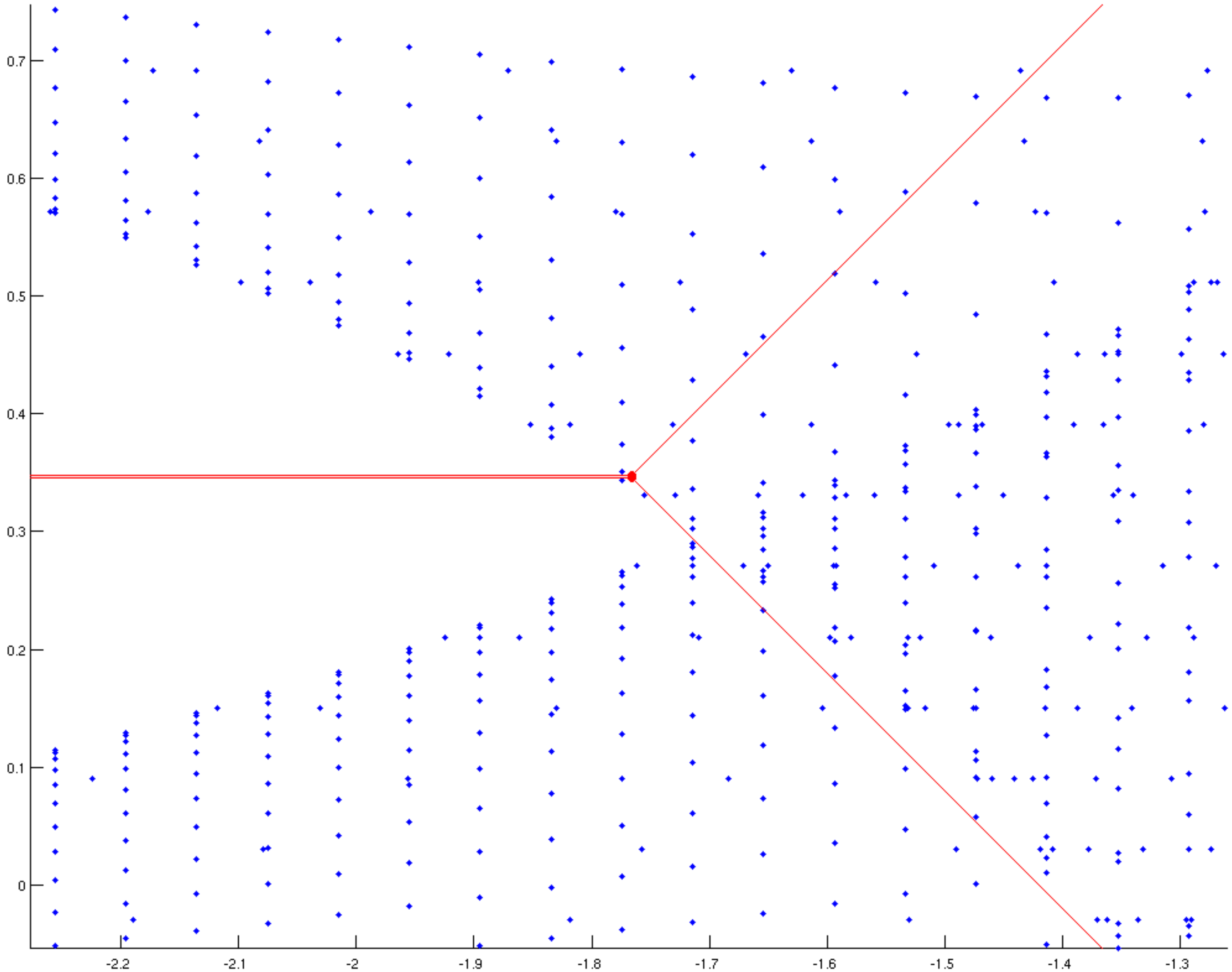


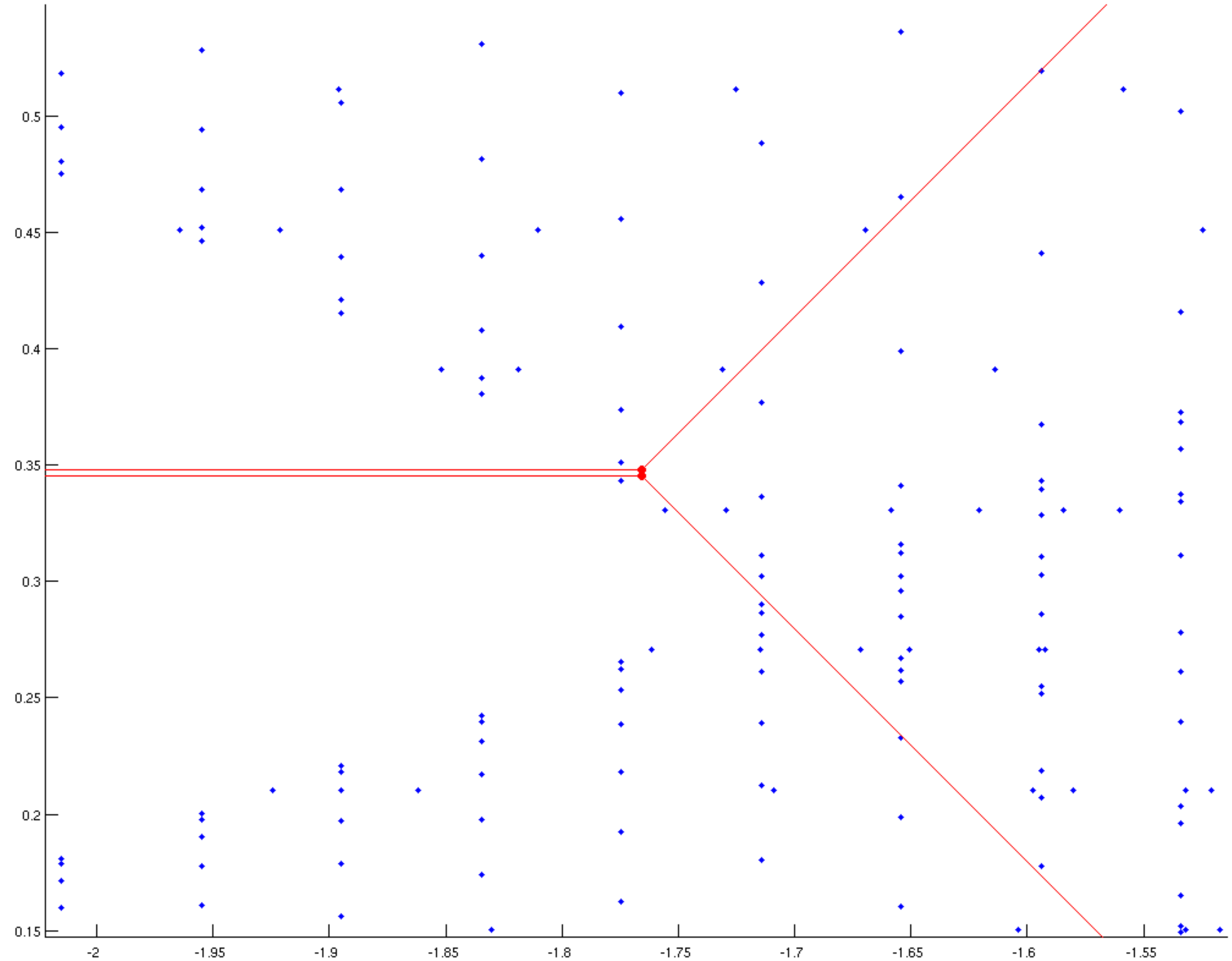
$$f(x_1, x_2) = 50x_1^3 + 83x_1^2x_2 + 24x_1x_2^2 + x_2^3 + 392x_1^2 + 414x_1x_2 + 50x_2 - 28x_1 + (50\sqrt{2} + .1)x_2 - 100$$



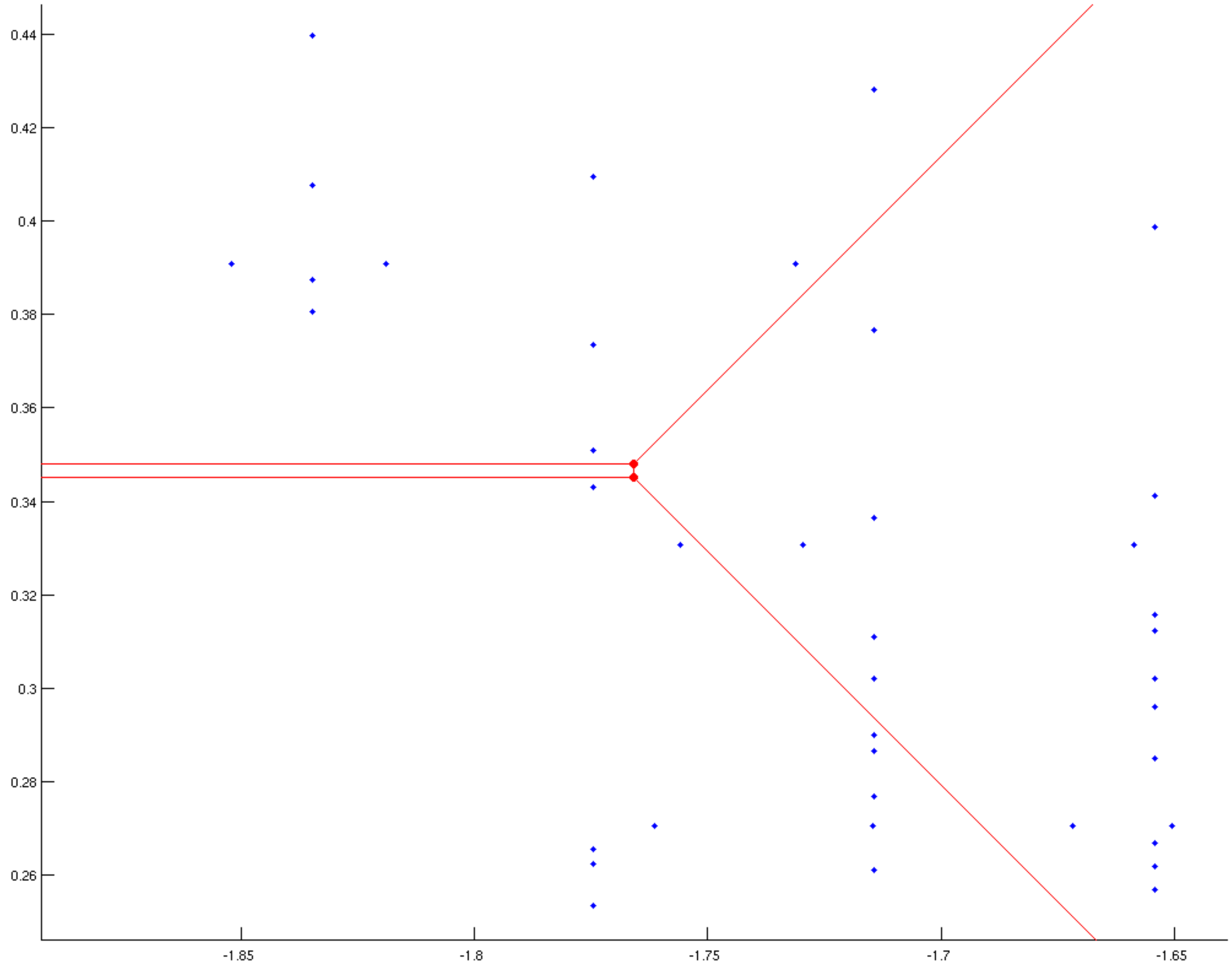
$$f(x_1, x_2) = 50x_1^3 + 83x_1^2 x_2 + 24x_1 x_2^2 + x_2^3 + 392x_1^2 + 414x_1 x_2 + 50x_2 - 28x_1 + (50\sqrt{2} + .1)x_2 - 100$$



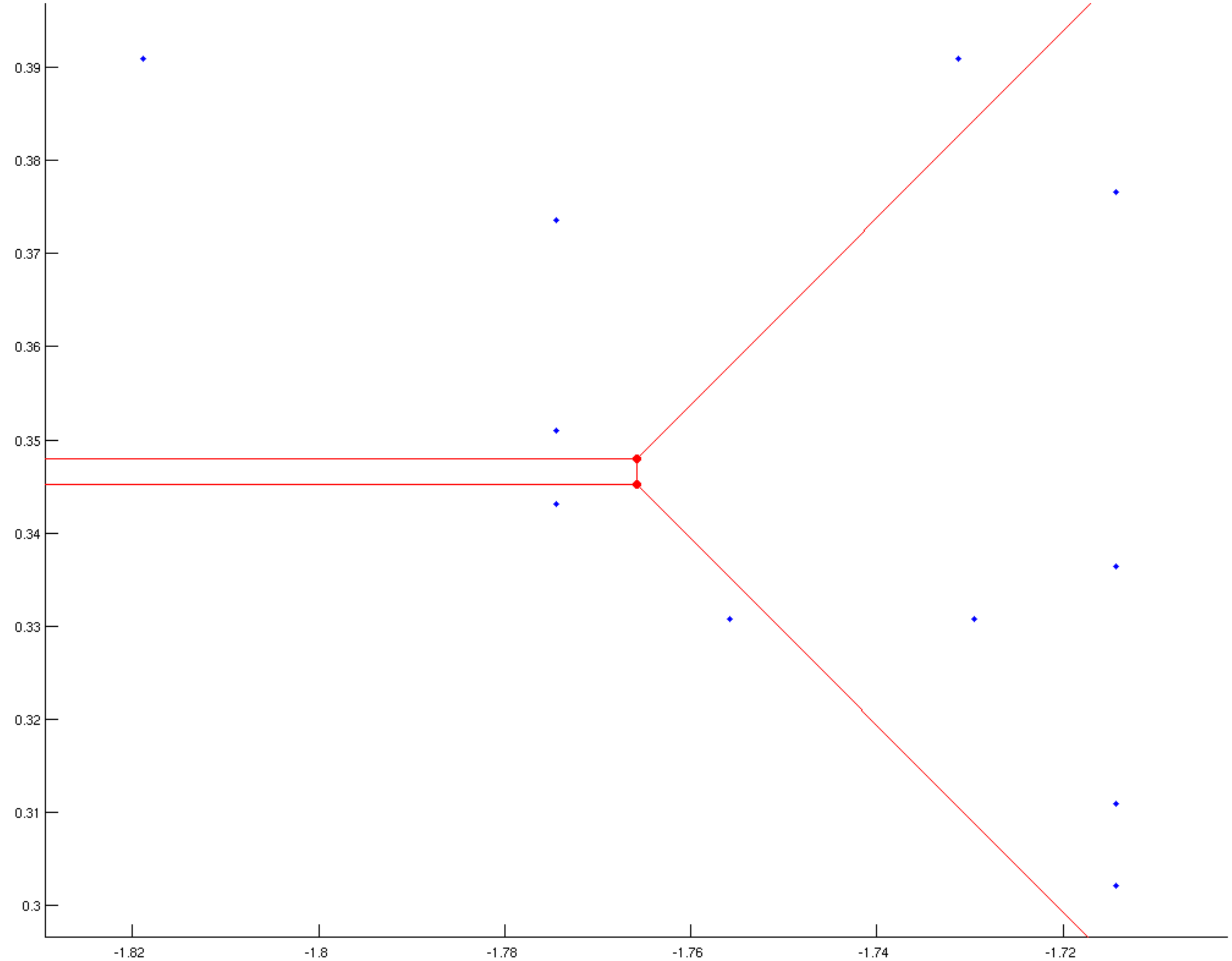
$f(x_1, x_2) = 50x_1^3 + 83x_1^2x_2 + 24x_1x_2^2 + x_2^3 + 392x_1^2 + 414x_1x_2 + 50x_2 - 28x_1 + (50\sqrt{2} + .1)x_2 - 100$



$f(x_1,x_2) = 50x_1^3 + 83x_1^2x_2 + 24x_1x_2^2 + x_2^3 + 392x_1^2 + 414x_1x_2 + 50x_2 - 28x_1 + (50\sqrt{2} + .1)x_2 - 100$

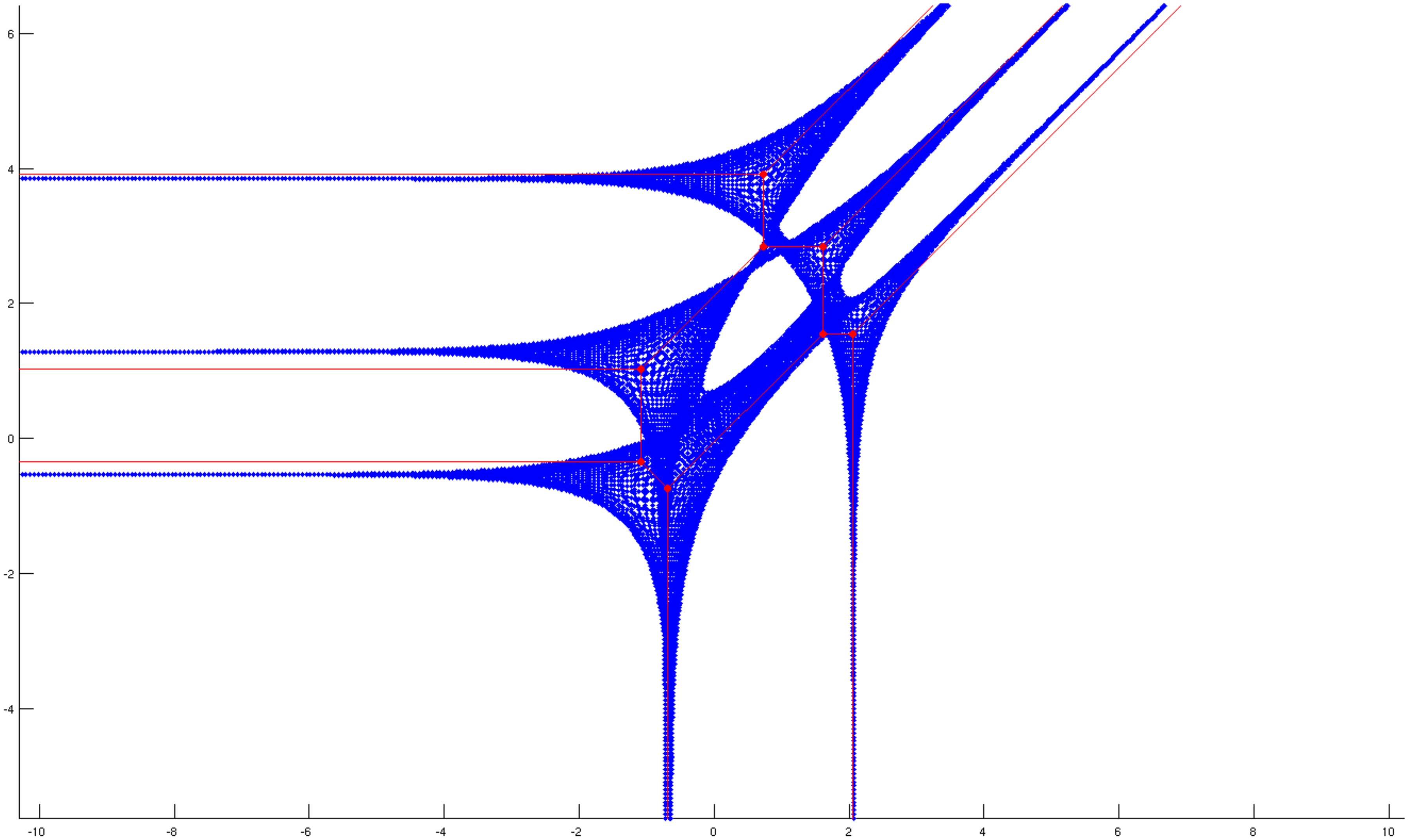


$f(x_1, x_2) = 50x_1^3 + 83x_1^2 x_2 + 24x_1 x_2^2 + x_2^3 + 392x_1^2 + 414x_1 x_2 + 50x_2 - 28x_1 + (50\sqrt{2} + .1)x_2 - 100$

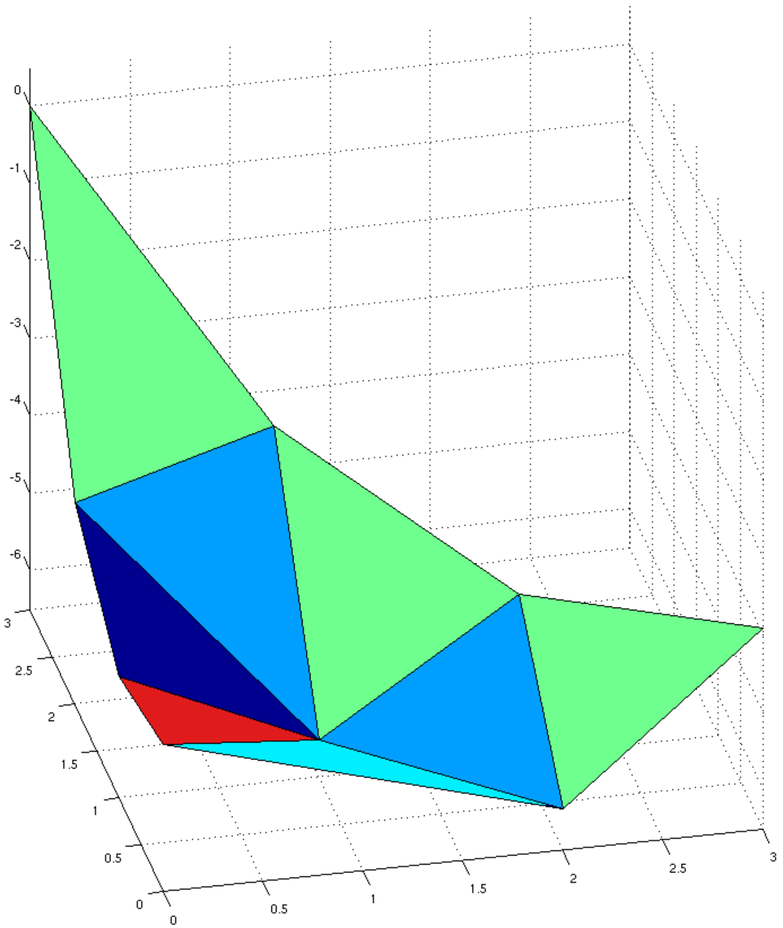


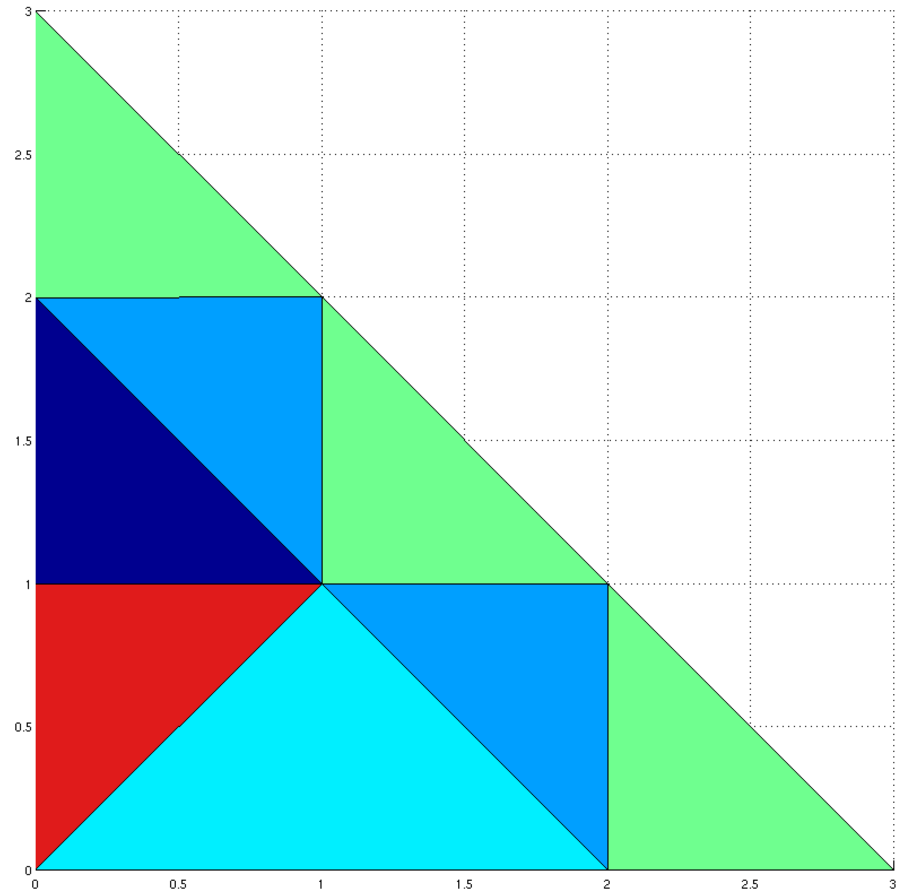
$$f(x_1, x_2) = 50x_1^3 + 83x_1^2 x_2 + 24x_1 x_2^2 + x_2^3 + 392x_1^2 + 414x_1 x_2 + 50x_2 - 28x_1 + (50\sqrt{2} + 70)x_2 - 100$$

$$f(x_1,x_2) = 50x_1^3 + 83x_1^2x_2 + 24x_1x_2^2 + x_2^3 + 392x_1^2 + 414x_1x_2 + 50x_2 - 28x_1 + (50\sqrt{2}+70)x_2 - 100$$

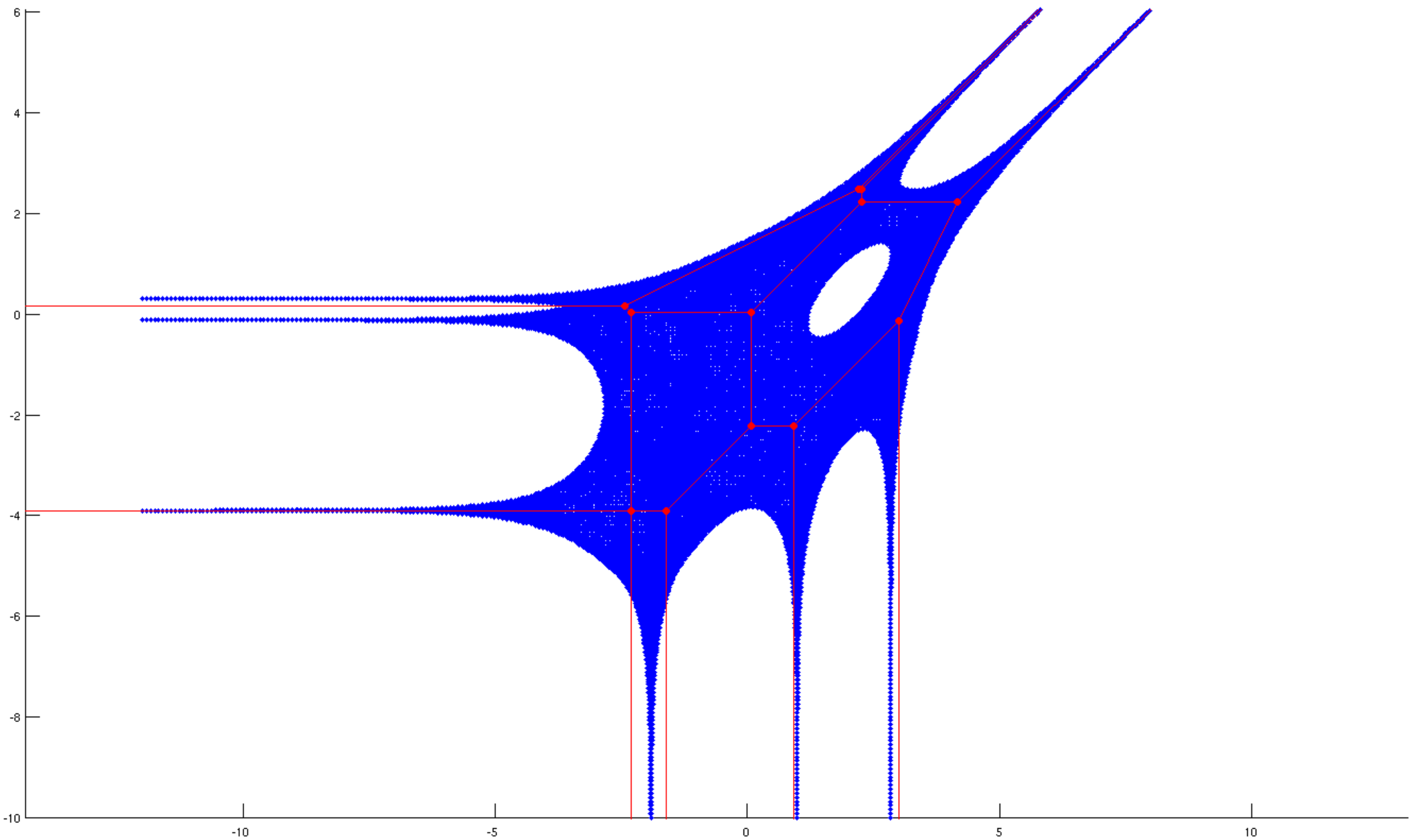


$$f(x_1, x_2) = 50x_1^3 + 83x_1^2x_2 + 24x_1x_2^2 + x_2^3 + 392x_1^2 + 414x_1x_2 + 50x_2 - 28x_1 + (50\sqrt{2} + 70)x_2 - 100$$





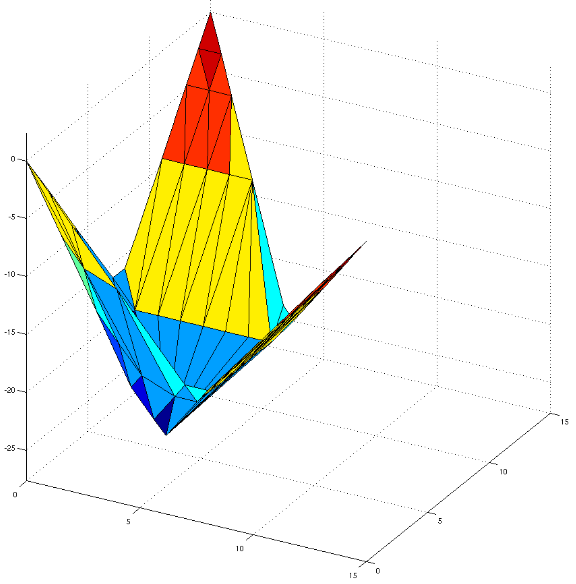
$$f(x_1,x_2) = x_1^4 + x_1^3x_2 + 50x_1^2x_2^2 + 40x_1x_2^3 + 30x_2^4 + 20x_1^3 + 460x_1^2x_2 + 480x_1x_2^2 + 10x_2^3 + 50x_1^2 - 500x_1x_2 + 40x_2^2 + 10x_1 + 50x_2 + 1$$

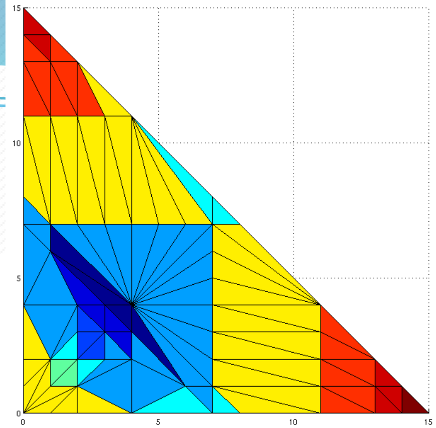


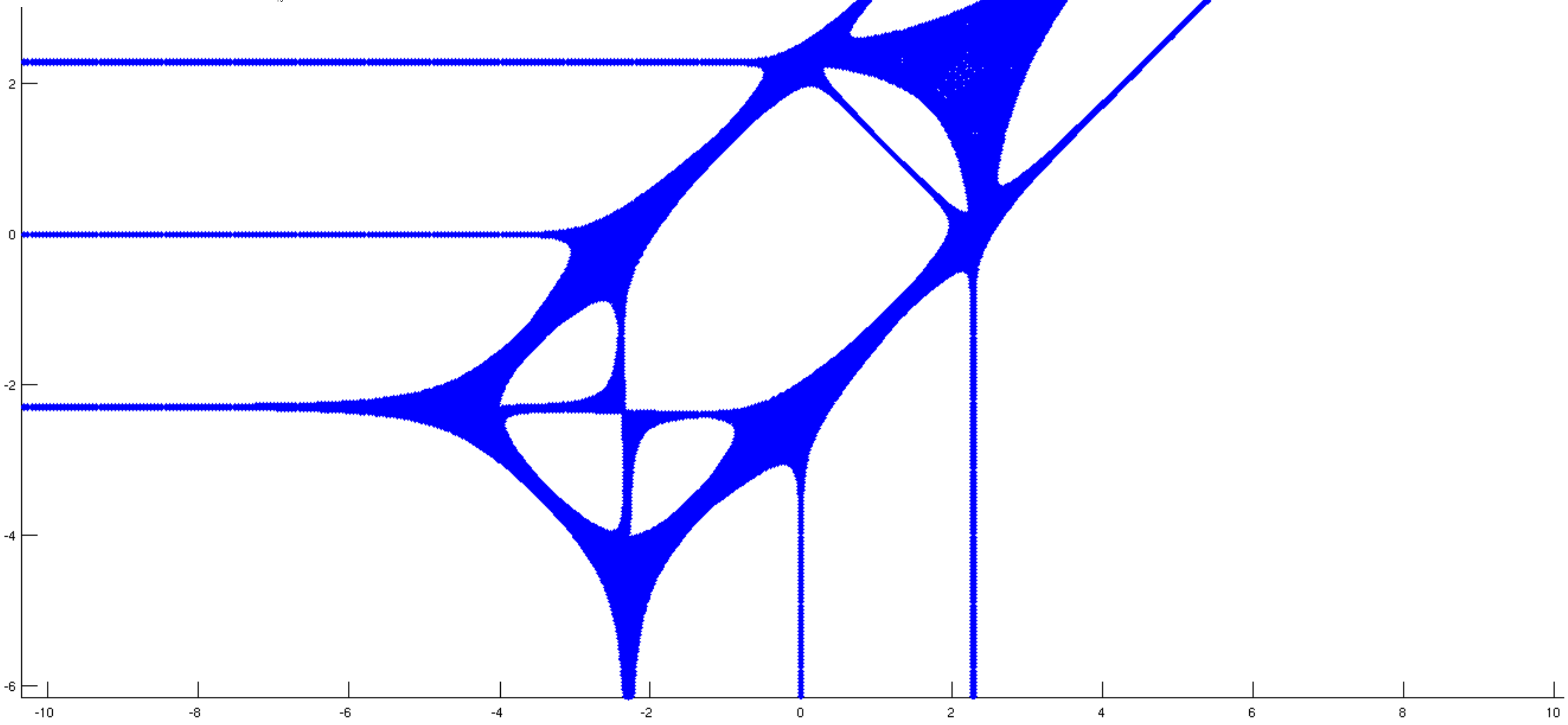
The Point

- For $f(x) = \sum_{i=1}^{t} c_i x^{a_i}$ where $a_1, \ldots, a_t \in \mathbb{Z}^2$:

$$\Delta(-\mathrm{Amoeba}(f), \mathrm{Trop}(f)) \leq \log(t-1)$$



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End of slide show, click to exit