1. Find

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}. \]

This is a telescoping series. \(1/(n+1)(n+2)) = 1/(n+1) - 1/(n+2),\)
so \(1/(n(n+1)(n+2)) = 1/n(1/(n+1) - 1/(n+2)) = 1/n - 1/(n+1) -
(1/2)((1/n) - 1/(n+2)).\) Now summing the first piece of this gives 1/1,
while with the second piece, the first two terms survive untelescoped so
we have a contribution of \(-(1/2)(1 + 1/2) = -3/4.\) Since \(1 - 3/4 = 1/4,\)
the answer to the question is 1/4.

2. Lake Cony has a radius of 1000 meters and fills a conical depression 100
meters deep. Water masses 1000 kg per cubic meter, and the acceleration
due to gravity is 9.81 meters/second^2. A joule is the energy needed to
accelerate a mass of 1 kilogram to a speed of 1 meter per second. Find
the energy (expressed in joules) needed to pump lake Cony dry.

This is a method-of-disks integral problem. The disk that is \(x\) meters
off the bottom of the lake has a radius of 10\(x\) meters, and an area of
100\(\pi x^2\). It must be raised \((100 - x)\) meters. Thus we have \(\int_{x=0}^{100} 100\pi x^2(100 - x) \, dx\)
cubic-meter-meter lifts of work to do. To lift one cubic meter of water
one meter is to move 1000 kgs with a force of 9.81 newtons each, up
one meter, and that takes 9810 joules. Grinding out the details gives
8.175\(\pi E12\) joules.

3. Let \(f(x) = \sum_{n=0}^{\infty} a_n x^n,\) where \(a_0 = 1, \ a_1 = 1/2, \ a_2 = -1/8, \ a_3 = 1/16, \ a_4 = -5/128,\) and in general, for \(n \geq 1, \ a_n = -(n - 3/2)a_{n-1}/n.\)

(a) Find \(a_5.\) The rule specifying the general case gives \(a_5 = -(5 - 3/2)a_4/5,\) and \(a_4 = -5/128,\) so \(a_5 = -(5 - 3/2)(-5/128)/5 = 7/256.\)

(b) Multiply out \(f(x) \cdot f(x)\) at least to the \(x^4\) term and then take an
informed guess at a simple formula for \(f(x)^2.\) This would amount to
expanding \((1 + x/2 - (1/8)x^2 + (1/16)x^3 - 5/128x^4 + \cdots)^2\) and
this multiplies out to \(1 + x + 0x^2 + 0x^3 + 0x^4 + ?x^5 + \cdots.\) (The coefficient
on \(x^4\) in the expansion is \(2 \cdot (-5/128 + (1/16) + 1/2) + (-1/8)^2 = 0,\)
and the others are easier.) Guess: \(f(x)^2 = 1 + x.\)

(c) Prove your guess. The series for \(f(x)\) is the Taylor’s series expansion
for \((1 + x)^{1/2}\) about \(x = 0\) because the \(n\)th derivative at zero of
\((1 + x)^{1/2}\) is the product of \((1/2 - j)\) over \(j\) from 0 to \(n - 1,\) and
that’s equivalent to the product of \((3/2 - k)\) over \(k\) from 1 to \(n,\) and
then we have to divide by \(n!\) to get the coefficient in the Taylor’s
series. This product obeys exactly the recursive rule given in the
problem, relating \(a_n\) to \(a_{n-1},\) because to extend the product by one
step from \( n - 1 \) to \( n \) we multiply by \( n - 3/2 \) and then the \( n! \) brings in a factor of \( 1/n \).

4. Suppose \( g(x) \) is continuous and differentiable everywhere, and \( g''(x) > 0 \) for all \( x \). Let \( h(x) = \int_0^x g(t) \, dt \).

(a) Sketch a few possibilities for the graphs of \( g(x) \) and the corresponding \( h(x) \).

(b) Prove that the graph of \( h(x) \) can cross the \( x \)-axis at most three times. Between any two crossings by \( h(x) \) of the \( x \)-axis, there must be a point at which the derivative of \( h \), which is \( g \) is zero. (Rolle’s theorem). But \( g \) is concave up because its second derivative is positive, so \( g' \) is strictly increasing. Thus there can be at most one place at which \( g' = 0 \). Between any two zeros of \( g \) there must be one zero of \( g' \), so \( g \) can have at most two places where it’s zero. That means at most three crossings of the \( x \)-axis by \( h \), as required.

5. For \( n \geq 1 \), let \( f_n(x) = nx e^{-nx^2} \). The graph of \( f_1(x) \) is shown:
(a) Sketch the functions $f_1(x)$, $f_2(x)$, and so on, all on the same graph.

(b) For $x > 0$, find $\lim_{n \to \infty} f_n(x)$. That’s zero. Using L’Hospital’s rule with $n$ as our variable, we have

$$
\lim_{n \to \infty} \frac{nx}{\exp(nx^2)} = \lim_{n \to \infty} \frac{(d/dn)(nx)}{(d/dn)\exp(nx^2)} = \lim_{n \to \infty} \frac{x}{x \infty} = 0.
$$

(c) Find $\int_{x=0}^{x=\infty} f_n(x) \, dx$. With the change of variable $u = nx^2$, $du = 2nx \, dx$ each of these integrals becomes $\int_{u=0}^{\infty} (1/2)e^{-u} \, du = 1/2$. All the integrals evaluate to $1/2$. The moral of the story is that the integral of the limit need not be equal to the limit of the integrals.