Properties of Wronski map of Grassmannians of spaces of solutions of linear differential operators

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Wronskian of a space of functions

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\[ \text{Wr}(f_1(t), f_2(t), \ldots, f_m(t)) := \det \begin{pmatrix}
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    f'_1(t) & f'_2(t) & \cdots & f'_m(t) \\
    \vdots & \vdots & \ddots & \vdots \\
    f^{(m-1)}_1(t) & f^{(m-1)}_2(t) & \cdots & f^{(m-1)}_m(t)
\end{pmatrix} \]

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Change of basis \(\rightarrow\) multiplication of Wronskian by a constant
If $\Lambda \in \text{Gr}_m(\text{Pol}_{m+p-1})$ is an $(m + p)$–dimensional space of univariant polynomials of degree at most $m + p - 1$, we will have

$$\text{Wr} : \text{Gr}_m(\text{Pol}_{m+p-1}) \longrightarrow \mathbb{P}(\text{Pol}_{mp})$$

Schubert in 1886 showed that the Wronski map is surjective and the general polynomial in $\mathbb{P}(\text{Pol}_{mp})$ has

$$\#_{m,p} = \frac{1!2! \ldots (p-1)! \cdot (mp)!}{m!(m+1)! \ldots (m + p - 1)!}$$

preimages.

$\text{Pol}_{m+p-1}$ is a space of solution of $x^{(m+p)} = 0$.

Natural question: What about the preimage of Wronski map for a space of solution of a general homogeneous linear differential operator?
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Natural question: What about the preimage of Wronski map for a space of solution of a general homogeneous linear differential operator?
Let $V_L$ be the space of solution of $Lx = 0$.

$W_\text{r} : \text{Gr}_m(V_L) \rightarrow \mathbb{P}(C^\infty)$.

- Wronski map on $\text{Gr}_m(V)$ is *strongly non-injective* if the preimage of a general point in the image contains more than one point.
- Wronski map on $\text{Gr}_m(V)$ is *essentially injective* if the preimage of a general point in the image contains only one point.
\[ Lx = x^{(2m)}(t) + a_{2m-1}(t)x^{(2m-1)}(t) + \ldots + a_0(t)x(t) \]

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Natural generalization

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Case of Self-adjoint operator

$L$ is called **self-adjoint** if

$$
\int_a^b L u \, v \, dt = \int_a^b u \, L v \, dt + A_{a,b}(u,v)
$$

where $A_{a,b}(u,v) = \sigma_b(u,v) - \sigma_a(u,v)$.

The restriction of $\sigma$ to the space of solution $V_L$ is a skew symmetric form on $V_L$.

**Proposition**

(Hein-Sottile-Zelenko, 2012, for $Lx = x^{(2m)}$; Zelenko for general self-adjoint $L$) If $L$ is self-adjoint, then $\text{Wr}(\Lambda) = \text{Wr}(\Lambda^\perp)$ where $\Lambda^\perp$ is the skew-symmetric complement w.r.t the form $\sigma \Rightarrow$ strong noninjectivity

**Question:** Is there a non-self adjoint operator for which the corresponding Wronski map is strongly non-injective?
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Geometric setting

Let $C(t)$ be a natural curve in $\mathbb{PV}^*$,

$$C(t) := \{p \in V^* : \langle p, x(\cdot) \rangle = 0 \quad \forall x(\cdot) \in V, \text{ such that } x(t) = 0\},$$

Let $c(t) \subset V^*$ such that $C(t) = \text{span}\{c(t)\}$,

$$C^{(i)}(t) := \text{span} \left( c(t), c'(t), \ldots, c^{(i)}(t) \right)$$

Plücker embedding $\mathbb{P}^X_m : Gr_m(X) \to \mathbb{P}(\wedge^m X) :$

$$\text{span}(x_1, \ldots, x_m) \to x_1 \wedge x_2 \ldots \wedge x_m$$
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\]

**Plücker embedding** \( \mathbb{P}^1_{m} : Gr_m(X) \rightarrow \mathbb{P}(\wedge^m X) : \)

\[
\text{span}(x_1, \ldots, x_m) \rightarrow x_1 \wedge x_2 \ldots \wedge x_m
\]
Wronski map and Plücker embedding

\[ S_L := \text{span}_t \text{Pl}^V_m(C^{(m-1)}(t)) \subset \mathbb{P}(\wedge^m V^*) \]

\[ A_L := \{ \omega \in (\wedge^m V^*)^* = \wedge^m V : \omega|_{S_L} = 0 \} = S_L^\perp. \]

Let \( \pi : \wedge^m V \to \wedge^m V/A_L \) be the canonical projection and

\[ \widetilde{Wr} := \pi \circ \text{Pl}^V_m : \text{Gr}_m(V) \to \wedge^m V/A_L. \]

**Proposition**

*Injectivity properties of \( Wr \) are the same as of \( \widetilde{Wr} \).*

**Proposition**

*If \( S_L = \mathbb{P}(\wedge^m V^*) \) or, equivalently \( A_L = 0 \) \( \Rightarrow \) \( Wr \) is injective.*
If \( m = 2 \), \( \dim A_L \geq 1 \Leftrightarrow \) there \( \exists \) a 2-form (symplectic form) \( \omega \) s.t. 
\[ \omega|_{C(m-1)(t)} = 0 \Leftrightarrow L \text{ is self-adjoint.} \]

What about \( m = 3 \)?

If \( L \) is self-adjoint, then \( \dim A_L \geq 6 \).
If \( L \) is trivial \( Lx = x^{(6)} \), then \( \dim A_L = 10 \) (iff).

The method is to study classical injectivity properties of \( \pi_K \circ \text{Pl}_m \)
where \( \pi_K : \wedge^m V \to \wedge^m V/K \) with \( \dim K = 1 \), \( K \subset A_L \).

\( \text{Gl}_{2m}(V) \) acts on \( \wedge^m V \), or to be specific, \( \mathbb{P}(\wedge^m V) \).

It is enough to consider this problem for a representative of each orbit under this action.
Reduction to the study of orbits

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Classification of orbits

Image of Plücker embedding = Plücker manifold. For $m = 3$, there are 5 different orbits under this action

- $\omega_0 = e_{123} + e_{456}$: open orbit $O_0$;
- $\omega_1 = e_{126} + e_{135} + e_{234}$: codim 1 orbit $O_1$, the closure is tangential variety of Plücker manifold;
- $\omega_5 = e_1 \wedge (e_{23} + e_{45})$: codim 5 orbit $O_5$, the union of lines connecting two points in Plücker manifold such that the 3-planes corresponding to these two points intersect in a line;
- $\omega_{10} = e_{123}$: codim 10 orbit $O_{10}$, Plücker manifold;
- $\omega_{20} = 0$. 
Results

- If $K \subset O_0 \rightarrow$ one pair of 3-planes with the same Wronskian.
- If $K \subset O_1 \rightarrow$ classically injective.
- If $K \subset O_5 \rightarrow$ 4-parametric family of pairs of 3-planes with same Wronskian.

**Theorem**

*If* $\dim A_L < 10$ and $\dim A_L \cap O_5 \leq 5$, *then* $\text{Wr}$ *is essentially injective.*

**Corollary**

*If* $\dim A_L \leq 5$, *then* $\text{Wr}$ *is essentially injective.*

**Theorem**

*If* $\dim A_L = 6$ and $\text{Wr}$ *is strongly non-injective, then* $L$ *is self-adjoint.*
Thanks for your attention.