ZEROS OF THE EISENSTEIN SERIES

RACHAEL WOOD

1. Introduction

It has been proved by Rankin and Swinnerton-Dyer [RS] that for the Eisenstein Series with $2k \geq 4$, the zeros of $E_{2k}(\tau)$ in the fundamental domain lie on the circle $|\tau| = 1$. This theorem has no parallel with respect to quasimodular forms. In fact, very little is known about the zeros of quasimodular forms. The Eisenstein Series of weight 2 is a quasimodular form. It is known that the Eisenstein series has infinitely many zeros within the half-strip of the complex plane [BS]. However, apart from this fact not much is known about the location of these zeros. This paper will further investigate various properties of these zeros and the equivariant function $h(z)$.

2. Background

To begin, we would like to familiarize the reader with some terminology. The Fundamental Domain, denoted by $D$, is given by,

$$D = \{ z \in \mathbb{H} : |z| \geq 1 \text{ and } -\frac{1}{2} \leq x \leq \frac{1}{2} \}.$$

The half-strip, denoted by $G$, is given by,

$$G = \{ z \in \mathbb{H} : -\frac{1}{2} \leq x \leq \frac{1}{2} \}.$$

Let $SL_2(\mathbb{Z})$ be the set of matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$.

We will now introduce the Eisenstein Series of weight $2k$ which has the Fourier expansion

$$E_{2k}(z) = 1 + \gamma_{2k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)e^{2\pi inz},$$

where

$$\gamma_{2k} = (-1)^k \frac{4k}{B_k},$$

$B_k$ is the $k$-th Bernoulli number, and $\sigma_{2k-1}(n) = \sum_{a|n} a^{2k-1}$.

When $k \geq 2$, $E_{2k}(z)$ is a modular form for $SL_2(\mathbb{Z})$ which means that it is holomorphic on $\mathbb{H}$, including $\infty$, and it satisfies the relations

$$f(z) = (cz + d)^{-2k}f\left(\frac{az + b}{cz + d}\right) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

This is equivalent to

$$f(z + 1) = f(z) \quad \text{and} \quad f\left(\frac{-1}{z}\right) = z^{2k}f(z).$$
There are no non-zero modular forms of weight 2 for $SL_2(\mathbb{Z})$. When $k = 1$, the function $E_2(z)$ defined by its Fourier expansion

$$E_2 = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)e^{2\pi inz}$$

is not a modular form. Rather, it is called a quasimodular form, which satisfies the relation

$$E_2\left(\frac{az + b}{cz + d}\right) = (cz + d)^2 E_2(z) - \frac{6}{\pi} ic(cz + d).$$

3. Zeros of $E_2(z)$

Basraoui and Sebbar [BS] showed that $E_2(z)$ has infinitely many zeros in $G$, none of which exist in $D$. However, very little is known about the actual location of these zeros. We used Mathematica to numerically solve the equation $E_2(z) = 0$ for $y \geq \varepsilon$ for various values of $\varepsilon$.

![Figure 3.1. $E_2(z) = 0$ for $y > .001$.](image)

As we were examining the data, we noticed that for $z = x + iy$ such that $E_2(z) = 0$, $\text{Re}(z)$ is very close to a rational number with a small denominator. Indeed, when we took
our data points and applied a rational approximation out to the 4th decimal place, we found that all the rational numbers within \( G \) were represented to a certain limit that increases as \( \epsilon \) gets closer to 0. We will display a small amount of our output for the reader to see. Note: Although we are showing both the \( x,y \) coordinates, we are only interested in the \( x \)-coordinate.

\((-0.5, 0.13091903039678807)\) is equal to \(-\frac{1}{2}\)

\((-0.3333258907443707, 0.0581819236539682)\) is very close approximation to \(-\frac{1}{3}\)

\((-0.24999517436865332, 0.03272491502484815)\) is a very close approximation to \(-\frac{1}{4}\)

\((-0.1999706592659725, 0.020942992285893466)\) is a very close approximation to \(-\frac{1}{5}\)

\((-0.400001820482515, 0.020946451273604345)\) is a very close approximation to \(-\frac{2}{5}\)

And this pattern continues for all the zeros of \( E_2(z) \) where \( y > 0.001 \). When we approximate these zeros, we generate a list of rational numbers. What you see below is just a small sample but is indicative our results. Notice that every rational number (within \( G \)) appears in the list out to a certain denominator. In this case, we stop our output at \( \frac{3}{10} \), but be assured this pattern continues.

\[ 0, -\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, -\frac{2}{5}, -\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, -\frac{3}{7}, -\frac{1}{7}, \frac{3}{7}, \frac{1}{7}, -\frac{4}{8}, -\frac{1}{8}, \frac{2}{8}, \frac{1}{8}, \frac{4}{9}, -\frac{2}{9}, -\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, -\frac{3}{10}, -\frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \ldots \]

4. Properties of \( h(z) \)

We now introduce the equation for \( h(z) \), which is defined by

\[ h(z) = z + \frac{6}{i\pi E_2(z)}. \]

This function \( h(z) \) is equivariant, which means that for \( z \in \mathbb{H} \) and \( \alpha \in SL_2(\mathbb{Z}) \), then

\[ h(\alpha z) = \alpha h(z). \]

See Sebbar-Sebbar [SS] for some properties of \( h \). Also note that \( h(z_0) = \infty \) is equivalent to \( E_2(z_0) = 0 \).

Proof of (4.1). Let \( \alpha \in SL_2(\mathbb{Z}) : \alpha = (a \ b) \) and \( z \in \mathbb{H} \)

Consider \( h(\alpha z) = \alpha z + \frac{6}{\pi i E_2(\alpha z)} \). By the transformation properties of \( E_2(\alpha z) \) and of \( \alpha \) we have,

\[ h(\alpha z) = \frac{az + b}{cz + d} + \frac{6}{\pi i [(cz + d)^2 E_2(z) + \frac{6}{\pi i} c(cz + d)]} = \frac{1}{cz + d} \left[ (az + b) + \frac{6}{\pi i E_2(z)} \right] \]
Proof. First, recall that for \( \alpha \in SL_2(\mathbb{Z}) \), \( ad - bc = 1 \).

Then
\[
h(\alpha z) = \frac{1}{cz + d} \left[ (az + b)(cz + d) + (az + b)\left(\frac{6c}{\pi i E_2(z)}\right) + \frac{6}{\pi i E_2(z)}(ad - bc) \right]
\]
\[
h(\alpha z) = \frac{1}{cz + d} \left[ (az + b)(cz + d) + a\left(\frac{6c}{\pi i E_2(z)}\right) \right]
\]
\[
h(\alpha z) = \frac{a(z + \frac{6}{\pi i E_2(z)}) + b}{c(z + \frac{6}{\pi i E_2(z)}) + d} = \frac{a(h(z)) + b}{c(h(z)) + d} = \alpha h(z)
\]
\[
\square
\]

Next we state a variation of Lemma 3.4 of Balasubramanian-Gun [BG], who worked with \( g(z) = 1/h(z) \).

**Theorem 4.1.** If \( E_2(z_0) = 0 \) then \( h(\gamma z_0) = \frac{a}{c} \) for \( \gamma = (\frac{a}{c}, \frac{b}{d}) \). Conversely, if \( h(\tau_0) = \frac{a}{c} \) with coprime \( a, c \), then \( E_2(\gamma^{-1}\tau_0) = 0 \) for \( \gamma = (\frac{a}{c}, \frac{b}{d}) \).

**Proof.** Consider the case when \( E_2(z_0) = 0 \) (so \( h(z_0) = \infty \)), and let \( z = \gamma z_0 \). Note that \( \gamma \infty = \frac{a}{c} \). Then
\[
h(\gamma z_0) = \gamma h(z_0) = \gamma \infty = \frac{a}{c}.
\]
Conversely, suppose \( h(\tau_0) = \frac{a}{c} \). Then
\[
h(\gamma^{-1}\tau_0) = \gamma^{-1}h(\tau_0) = \gamma^{-1}\frac{a}{c} = \infty,
\]
so \( E_2(\gamma^{-1}\tau_0) = 0 \).

Since \( h(z) \) is rational only when \( E_2(z) = 0 \), by graphing \( \text{Im}(h(z)) = 0 \), all of the solutions to \( E_2(z) = 0 \) will be plotted along with some other values. The graphs of \( \text{Im}(h(z)) = 0 \) are placed at the end of the paper, but the images will be discussed here.

The “almost-circular” shapes in Figures 4.1 and 4.2 can be shown to be nearly circular. By applying all the elements of \( SL_2(\mathbb{Z}) \) to the curve that satisfies \( \text{Im}(h(z)) = 0 \) in \( \mathbb{D} \), our resulting images are the nearly circular shapes that we see below \( \mathbb{D} \). Therefore, our curve in \( \mathbb{D} \) is the generating curve for all the solution curves in \( \mathbb{H} \). The curve generated by \( \text{Im}(h(z)) = 0 \) in \( \mathbb{D} \) can be bounded above and below by straight lines. By this fact, we know the curves below \( \mathbb{D} \) are very close to circles. Because as \( SL_2(\mathbb{Z}) \) translates these 2 lines and our curve, the image is two perfect circles with the translated curve sitting in-between.

These graphs prompted us to ask the question: What would happen if we transformed the zeros of \( E_2 \) into \( \mathbb{D} \)? Recall, \( E_2(z) \) has no zeros in \( \mathbb{D} \). However, we can translate the coordinates of each of our zeros back into \( \mathbb{D} \) by applying different elements of \( SL_2(\mathbb{Z}) \) to each individual point. Figure 4.4 shows a sample of some of our zeros of \( E_2(z) \) which have been translated back into \( \mathbb{D} \). When we show the curve of \( \text{Im}(h(z)) = 0 \) which is in \( \mathbb{D} \) along
with our translated zeros, we have Figure 4.5. The fact that these lie on the same curve is a
expected consequence of Theorem 4.1 and the fact that the curve of \( \text{Im}(h(z)) = 0 \) in \( \mathbb{D} \)
is the generating curve for all values of curve of \( \text{Im}(h(z)) = 0 \). Indeed, if we translated all
of the zeros of \( E_2(z) \) back into \( \mathbb{D} \), the resulting image would be the same as Figure 4.3 (i.e.
they would trace out the curve of \( \text{Im}(h(z)) = 0 \) in \( \mathbb{D} \)).

**Theorem 4.2** (M. Young, R. Wood). *The real values of the function \( h(z) \) which occur in
the fundamental domain \( \mathbb{D} \) occur only in the small strip \( |y - 6/\pi| < .00028 \).

*Proof.* (Note: A more rigorous proof with actual bounds for the error terms is in progress)

Let \( z \in \mathbb{D} \) therefore, \( \sqrt{\frac{3}{2}} \leq y < \infty \).

\[
h(z) = (x + iy) + \frac{6}{\pi i E_2(x + iy)}
\]

\[
h(x + iy) = (x + iy) + \frac{6}{\pi i[1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n(x+iy)}]}
\]

\[
h(x + iy) \approx (x + iy) + \frac{6}{\pi i} \frac{1}{1 - 24 e^{2\pi i x} e^{-2\pi y} + \varepsilon}
\]

where \( \varepsilon \) is a negligible error term.

Recall, the Taylor Series for \( \frac{1}{1-u} = 1 + u + u^2 + u^3 + .... \) Therefore if we let

\[
u = 24(1)e^{2\pi i x} e^{-2\pi y} + \varepsilon
\]

we find that

\[
h(x + iy) \approx (x + iy) + \frac{6}{\pi i} [1 + u + u^2 + u^3 + ....].
\]

\[
h(x + iy) \approx (x + iy) + \frac{6}{\pi i} \left[ 1 + (24(1)e^{2\pi i x} e^{-2\pi y} + \varepsilon) + (24(1)e^{2\pi i x} e^{-2\pi y} + \varepsilon)^2 + .... \right]
\]

\[
\varepsilon_0 = \left[ (24(1)e^{2\pi i x} e^{-2\pi y} + \varepsilon)^2 + (24(1)e^{2\pi i x} e^{-2\pi y} + \varepsilon)^3 + (24(1)e^{2\pi i x} e^{-2\pi y} + \varepsilon)^4 + .... \right]
\]
is also another negligible error term for our purposes.

\[
h(x + iy) \approx (x + iy) + \frac{6}{\pi i} (1 + (24(1)e^{2\pi i x} e^{-2\pi y}))
\]

\[
h(x + iy) \approx x + iy + \frac{24*6}{\pi i} e^{-2\pi y} \left[ \cos 2\pi x + i \sin 2\pi x \right] + \left[ \text{small error} \right]
\]

\[
h(x + iy) \approx \left[ x + \frac{24*6}{\pi} \sin 2\pi x e^{-2\pi y} \right] + \left[ y - \frac{6}{\pi} - \frac{24*6}{\pi} \cos 2\pi x e^{-2\pi y} \right]
\]

If \( y > \frac{6}{\pi} + \frac{24*6}{\pi} e^{-2\pi y} + .... \), then \( h(x + iy) \) is not real.

If \( y < \frac{6}{\pi} - \frac{24*6}{\pi} e^{-2\pi y} + .... \), then \( h(x + iy) \) is not real.

For \( h(x + iy) \) to be real we need,

\[\frac{6}{\pi} - \frac{24*6}{\pi} e^{-2\pi y} \leq y \leq \frac{6}{\pi} + \frac{24*6}{\pi} e^{-2\pi y}\]
Let \( y = \frac{6}{\pi} + \delta \) where \( \delta \) is very small.

\[
\frac{6}{\pi} - \frac{24 \ast 6}{\pi} e^{-2\pi y} \leq \frac{6}{\pi} + \delta \leq \frac{6}{\pi} + \frac{24 \ast 6}{\pi} e^{-2\pi y}
\]

\[
-\frac{24 \ast 6}{\pi} e^{-2\pi (\frac{\delta}{\pi} + \delta)} \leq \delta \leq \frac{24 \ast 6}{\pi} e^{-2\pi (\frac{\delta}{\pi} + \delta)}
\]

\[
-\frac{24 \ast 6}{\pi} e^{-12} e^{-2\pi \delta} \leq \delta \leq \frac{24 \ast 6}{\pi} e^{-12} e^{-2\pi \delta}
\]

\[
-\frac{24 \ast 6}{\pi} e^{-12} \leq e^{2\pi \delta} \delta \leq \frac{24 \ast 6}{\pi} e^{-12}
\]

\[
e^{2\pi \delta} \approx 1 \text{ since } \delta \text{ is very small and } e^0 = 1
\]

\[
-\frac{24 \ast 6}{\pi} e^{-12} \leq \delta \leq \frac{24 \ast 6}{\pi} e^{-12}
\]

\[
|\delta| \leq \frac{24 \ast 6}{\pi} e^{-12} \approx .00028
\]

\[
|y - \frac{6}{\pi}| \leq \frac{24 \ast 6}{\pi} e^{-12} \approx .00028
\]

\[
\square
\]

Therefore, by Theorem 4.1 and Theorem 4.2, we can conclude that all of the zeros of \( E_2(z) \) can be translated back into \( \mathbb{D} \) and lie on the curve bounded above and below by

\[
|y - \frac{6}{\pi}| \leq \frac{24 \ast 6}{\pi} e^{-12} \approx .00028
\]

**References**


Figure 4.1. Graph of $\text{Im}(h(z)) = 0$

Figure 4.2. Zoomed in graph of $\text{Im}(h(z)) = 0$ below $\mathbb{D}$
Figure 4.3. Zoomed in graph of $\text{Im}(h(z)) = 0$ in $\mathbb{D}$

Figure 4.4. $SL_2(\mathbb{Z})$ Translated Zeros of $E_2(z) = 0$ into $\mathbb{D}$
Figure 4.5. Plot of Translated Zeros of $E_2(z) = 0$ and $Im(h(z)) = 0$ in $\mathbb{D}$