Visualizing $A$-Discriminant Varieties and their Tropicalizations

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Agenda

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- Approaches
- Important Concepts
- Amoeba
- Approximation of the Amoeba
- Goals
- Applications
Abel’s Theorem states that, for polynomials of degree 5 or higher, it is not possible to express the general solutions of a polynomial equation in terms of radicals.

This theorem points to the need for more general iterative algorithms that go beyond taking radicals.
Given $f(x) = x^4 - 2x^2 + 1$:

- $P_f(x) = (x^4 - 2x^2 + 1, 4x^3 - 4x, x^2 - 1, 0, 0)$
- $\sigma(P_f(-3)) = (1, -1, 1)$ and $\sigma(P_f(3)) = (1, 1, 1)$
- $V_f(-3) = 2$ and $V_f(3) = 0$

For $f$, the number of roots between -3 and 3 is 2.

When computing the Sturm Sequence for $f(x) = x^{317811} - 2x^{196418} + 1$, the polynomials needed to complete the computation have hundreds of thousands of digits.
Approaches: Classifying Polynomials

Two ways to classify the polynomial:

\[ f(x, y) = c_0x^3 + c_1x^2y^2 + c_2y^3 + c_3 \]

- Based on degree: \( f(x, y) \) is a cubic polynomial.

- Based on number of variables and terms \( f(x, y) \) is a bi-variate, 4 - nomial.

Using the second method can be useful when dealing with polynomials of high degree with few terms.
For each \((n + k)\)-nomial case, we have families of polynomials with the same exponents.

**Example: \(n + 3\) Case**

\[
f(x) = c_0x^3 + c_1x^2 + c_2x + c_3
\]

\[
g(x, y) = c_0x^6y^2 + c_1x^2y^{-7} + c_2x^2y^5 + c_3x + c_4y
\]

\(7x^3 + 1x^2 + 4x + 8\) and \(-23543x^3 + 12345x^2\) are in the same family.
Important Concepts: Support

- For each \((n + k)\)-nomial case, we have families of polynomials with the same exponents.
- Each family can be represented by its support.

**Definition**

Given \(f(x_1, x_2, \ldots, x_n) = c_1 x^{a_1} + c_2 x^{a_2} + \cdots + c_t x^{a_t}\) where \(t\) represents the number of terms, \(c_i \in \mathbb{C}, a_i \in \mathbb{Z}^n\)

\[
supp(f) = \mathcal{A} = \{a_1, \ldots, a_t\}
\]

**Example: \(n + 3\) Case**

\[f(x) = c_0 x^3 + c_1 x^2 + c_2 x + c_3\]
\[g(x, y) = c_0 x^6 y^2 + c_1 x^2 y^{-7} + c_2 x^2 y^5 + c_3 x + c_4 y\]

\[
supp(f) = [3 \ 2 \ 1 \ 0] \quad supp(g) = \begin{bmatrix} 6 & 2 & 2 & 1 & 0 \\ 2 & -7 & 5 & 0 & 1 \end{bmatrix}
\]
Important Concepts: $\Delta_{A}$ and $\nabla_{A}$

- For a given support, we can find the $A$-discriminant, $\Delta_{A}$.
- $\nabla_{A}$ refers to the zero set of $\Delta_{A}$.
- Each element in $\nabla_{A}$ represents a polynomial with degenerate roots (a root where the Jacobian determinant vanishes).

**Example**

Given $c_0x^2 + c_1x + c_2$

$A = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$

$\Delta_{A} = c_1^2 - 4c_0c_2$

$\nabla_{A}$ refers to the solution set of $c_1^2 - 4c_0c_2$

Because $(2, 4, 2)$ and $(1, 6, 9)$ are elements of $\nabla_{A}$, we know $2x^2 + 4x + 2$ and $x^2 + 6x + 9$ have degenerate roots.
Important Concepts: $\mathcal{A}$ - Discriminants

- We can plot $\nabla_{\mathcal{A}}$ in a dimension equal to the number of terms.
- The visualization represents every real polynomial in a family.
- Each point on the plot is a polynomial with degenerate roots.

Figure: Quadratic Case
Important Concepts: Parametrization

- The $\mathcal{A}$-discriminant polynomial can become difficult to calculate.

- We can find a parametrization to describe the solution set without solving for $\Delta_{\mathcal{A}}$.

- By taking the log of this parametrization, we obtain a visualization for understanding a family of polynomials, the amoeba.
The amoeba of any polynomial, $f$, is the log of the absolute value of the zero set of $f$.

To plot the $A$-discriminant amoeba, we find the zero set, $\nabla_A$, and plot $\log|\nabla_A|$.

We can create a visualization in a lower dimension by plotting the amoeba of the reduction of the polynomial.
Amoeba: Reduced $\mathcal{A}$-Discriminant Amoeba

With division and rescaling $f(x) = c_0 x^2 + c_1 x + c_2$ can be reduced to $x^2 + x + c$.

$\Delta_A = c_1^2 - 4c_0c_2$

$\overline{\Delta}_A = 1 - 4c$
Amoeba: Visualization

We can visualize the reduced $\mathcal{A}$-discriminant amoeba for $(n + 2), (n + 3)$ and $(n + 4)$-nomials.

The contour is the image of the real zero set of a polynomial under the $\text{Log} | \cdot |$ map.
Amoeba: Importance

- The complement of the amoeba is the finite disjoint union of open convex sets.
- These unbounded open convex sets are called outer chambers.
- The topology of the real zero set is constant in each outer chamber.

**Figure:** Amoeba($\Delta_A$): Polynomial of degree 31, 8 terms
Amoeba: Importance

- The topology of the real zero set is constant in each outer chamber.
- The zero sets of the polynomials within each chamber are isotopic.
Computing an amoeba can be inefficient. Instead, we can use an approximation to estimate where the amoeba and its chambers lie.

Chamber cones are used as an approximation of amoeba.

Chamber cones are used as an approximation of amoeba.
Approximations: Tropical $A$-Discriminant

- The tropical $A$-discriminant is the union of cones centered at the origin.

- The tropical $A$-discriminant can be found more quickly than the chamber cones.
Goals

- Create an algorithm to visualize the reduced $A$-discriminant amoeba for $(n + 4)$-nomials.

- Create an algorithm to compute the reduced tropical $A$-discriminant for $(n + 4)$-nomials.
Applications

- Polynomial models are used in: robotics, mathematics biology, game theory, statistics and machine learning.
- Certain problems in physical modeling involve solving systems of real polynomial equations.
- Many industrial problems involve sparse polynomial systems whose real roots lie outside the reach of current algorithmic techniques.
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Thank you for listening!