Effective Non-vanishing of Class Group \( L \)-Functions for Biquadratic CM Fields

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(Joint with Adrian Barquero-Sanchez and Emily Peirce)
Fix the following notation:

- $F$ is a number field.
- $d_F$ is the absolute value of the discriminant of $F$.
- $\mathcal{O}_F$ is the ring of integers of $F$.
- $\mathcal{O}_F^\times$ is the group of units of $\mathcal{O}_F$.
- $Cl(\mathcal{O}_F)$ is the ideal class group of $F$.
- $h_F$ is the class number.
- $R_F$ is the regulator of $F$.
- $\zeta_F(s)$ is the Dedekind zeta function.
- $\gamma_F$ is the constant term of $\zeta_F(s)$ at $s = 1$. 
Recall that:

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- $\mathcal{O}_F$ is a PID $\iff F$ has class number 1.
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Note. $\#\hat{G} = \#G$. 
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Class group $L$-functions

Definition

Given $\chi \in \widehat{Cl}(\mathcal{O}_F)$, we define the class group $L$-function by

$$L(\chi, s) = \sum_{C \in Cl(\mathcal{O}_F)} \chi(C) \zeta_F(s, C)$$

where

$$\zeta_F(s, C) = \sum_{0 \neq I \in C} N(I)^{-s}$$

is the partial zeta function.
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- One has a functional equation relating $L(\chi, s)$ to $L(\chi, 1 - s)$.
- The “central value” is $L(\chi, \frac{1}{2})$.
- We wish to determine whether $L(\chi, \frac{1}{2}) \neq 0$. 
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- Let $y = \text{Im}(z) = (y_1, \ldots, y_n)$ and $N(y) = \prod_{j=1}^{n} y_j$. 
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Let $N(\alpha + \beta z) = \prod_{j=1}^{n} (\sigma_j(\alpha) + \sigma_j(\beta)z_j)$ for $\alpha, \beta \in K$. 

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Definition
The *Hilbert modular Eisenstein series* is defined by

\[ E_K(z, s) = \sum_{0 \neq (\alpha, \beta) \in \mathcal{O}_K^2 / \mathcal{O}_K^x} \frac{N(y)^s}{|N(\alpha + \beta z)|^{2s}}, \quad z \in \mathbb{H}^n, \quad \text{Re}(s) > 1. \]
The average formula

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**Proposition**

For $\chi \in \text{Cl}(\mathcal{O}_E)$, we have

$$\frac{1}{h_E} \sum_{\chi \in \text{Cl}(\mathcal{O}_E)} L(\chi, s) = \left( \frac{2^n d_K}{\sqrt{d_E}} \right)^s \frac{1}{[\mathcal{O}_E^\times : \mathcal{O}_K^\times]} E_K(z_{\mathcal{O}_E}, s),$$

where $z_{\mathcal{O}_E} \in \mathbb{H}^n$ is a certain special point depending on $\mathcal{O}_E$. 
Theorem (B-S,P,Weber)

Let $d_1 > 0$ and $d_2 < 0$ be squarefree, coprime integers with $d_1 \equiv 1 \mod 4$ and $d_2 \equiv 2$ or $3 \mod 4$. Assume $K = \mathbb{Q}(\sqrt{d_1})$ has class number 1 and let $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Then if

$$|d_2| \geq (318310)^2 d_1 \exp \left\{ \sqrt{d_1} (\log(4d_1) + 2) \right\}$$

then there exists a character $\chi \in \widehat{Cl(O_E)}$ such that $L(\chi, \frac{1}{2}) \neq 0$. 