Lyme Disease: A Mathematical Approach

Antoine Marc & Carlos Munoz

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Overview

1. Biology of Lyme Disease
   - Borrelia burgdorferi
   - Ixodes scapularis
   - Hosts

2. Base Model
   - Compartmental Model
   - System of ODEs
   - Analysis of $R_0$
   - Simulations
   - Conclusion

3. Age-Structured Tick Class Model

4. Age-Structured Tick Class & Seasonality Model
   - System of ODEs
   - Analysis of $R_0$
   - Simulations
   - Conclusion
Why is Lyme Disease Important to Study?

- The shortening of Winter in the North led to the following:
- Warmer temperatures have been predicted to both enhance transmission intensity and extend the distribution of diseases such as malaria and dengue as well.
- Climate change may open up previously uninhabitable territory for arthropod vectors as well as increase reproductive and biting rates, and shorten the pathogen incubation period.
Borrelia Burgdoferi

Lyme Disease Bacteria
*Borrelia burgdorferi*

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Ixodes scapularis

- *Ixodes scapularis*, the black-legged tick
- Can be found throughout the country **including Texas**
- Take a blood meal every time they molt
- Once infected, they are infected for life
- Must attach for 36 hours to transmit the bacteria
- No vertical transmission
- Questing season is changing due to climate change
Biology of Lyme Disease
Base Model
Age-Structured Tick Class Model
Age-Structured Tick Class & Seasonality Model

Borrelia burgdorferi
Ixodes scapularis
Hosts

Figure of Life cycle:

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Lyme Disease: A Mathematical Approach
White footed Mouse, *Peromyscus leucopus*
Unidentified Alternate Host

![Raccoon](image1.png)
![Deer](image2.png)
![Bird](image3.png)
1. Recent data shows that ticks quest at two distinct heights
2. This will help narrow down the search for the other hosts
Compartmental Model for Base Model

\[ \begin{align*}
S_M & \quad \text{Susceptible Males} \\
I_M & \quad \text{Infected Males} \\
S_T & \quad \text{Susceptible Ticks} \\
I_T & \quad \text{Infected Ticks} \\
S_A & \quad \text{Susceptible Adults} \\
I_A & \quad \text{Infected Adults}
\end{align*} \]

\[ \begin{align*}
\rho_M N_M & \quad \text{Birth Rate} \\
\mu_M & \quad \text{Death Rate} \\
\beta_M S_M \frac{I_T}{N_T} & \quad \text{Transmission Rate} \\
\rho_T N_T & \quad \text{Birth Rate} \\
\mu_T & \quad \text{Death Rate} \\
\beta_T S_T \left( \alpha \frac{I_M}{N_M} + (1-\alpha) \frac{I_A}{N_A} \right) & \quad \text{Transmission Rate} \\
\rho_A N_A & \quad \text{Birth Rate} \\
\mu_A & \quad \text{Death Rate} \\
\beta_A S_A \frac{I_T}{N_T} & \quad \text{Transmission Rate}
\end{align*} \]
Base Model

\[
\frac{di_M}{dt} = \alpha \tilde{\beta}_M (1 - i_M) i_T - \mu_M i_M
\]

\[
\frac{di_T}{dt} = \tilde{\beta}_T (1 - i_T) \left( (\alpha i_M) + (1 - \alpha) i_A \right) - \mu_T i_T
\]

\[
\frac{di_A}{dt} = (1 - \alpha) \tilde{\beta}_A (1 - i_A) i_T - \delta_1 \mu_A i_A
\]
Base Model with Seasonality

\[
\frac{di_M}{dt} = \alpha \tilde{\beta} M (1 - i_M) i_T - \mu_M i_M \\
\frac{di_T}{dt} = \tilde{\beta} T (1 - i_T) \left( (\alpha i_M) + (1 - \alpha) i_A \right) - \mu_T i_T \\
\frac{di_A}{dt} = \delta_1 (1 - \alpha) \tilde{\beta} A (1 - i_A) i_T - \delta_1 \mu_A i_A
\]
middle $\frac{2}{3}:61-304$

$$\delta_1 = \begin{cases} 1 & \text{if } 61 \leq t \leq 304 \\ 0 & \text{Otherwise} \end{cases}$$
Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_M$</td>
<td>Birth rate of the mice into the susceptible class</td>
</tr>
<tr>
<td>$\rho_T$</td>
<td>Birth rate of the ticks into the susceptible class</td>
</tr>
<tr>
<td>$\rho_A$</td>
<td>Birth rate of the alternate host into the susceptible class</td>
</tr>
<tr>
<td>$\beta_M$</td>
<td>Contact Transmission Rate for the mice</td>
</tr>
<tr>
<td>$\beta_T$</td>
<td>Contact Transmission Rate for the Tick</td>
</tr>
<tr>
<td>$\beta_A$</td>
<td>Contact Transmission Rate for the alternate host</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Proportion of the ticks that have a preference for questing at lower heights</td>
</tr>
<tr>
<td>$\mu_M$</td>
<td>Death rate for the Mouse class</td>
</tr>
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Table: Table of Variables & Parameters for Base Model
Basic Reproduction Number of an Infection

The nondimensionalized system was reduced and rearranged into an equation for $R_0$, which determines whether or not there will be an epidemic.

1. If $R_0 < 1$ then the disease will eventually die out of the population.
2. If $R_0 = 1$ the disease remains at a constant level in the population.
3. If $R_0 > 1$ the level of disease in the population will increase until there is an epidemic.
\[ R_0 := \sqrt{\frac{\alpha^2 \beta_M \beta_T \mu_A + (1 - \alpha)^2 \beta_A \beta_T \mu_M}{\mu_M \mu_T \mu_A}} \]
Scenarios in which the overall $R_0 > 1$ and an epidemic will occur in the community.
Overall $R_0 > 1$ and an epidemic occurs
Scenarios in which the overall $R_0 > 1$ and an epidemic will occur in the community Cont.
Overall $R_0 < 1$ and the disease dies out
Adding an invading species with $R_0 > 1$ can increase the level of infection for the initial host.
Three Tick Stages

Figure A: Larva (A), nymph (B), adult male (C), adult female (D), and engorged adult female with eggs (E) of *I. scapularis*. Image courtesy of James Occi.
System of ODEs Modeling the Spread of Lyme Disease
System of ODEs Modeling the Spread of Lyme Disease

\[
\begin{align*}
\frac{di_M}{dt} &= \alpha \beta_M (1 - i_M) i_T - \mu_M i_M \\
\frac{di_{T_L}}{dt} &= \beta_T (1 - i_{T_L}) i_M - \eta_T i_{T_L} - \mu_T i_{T_L} \\
\frac{di_{T_{N,E}}}{dt} &= \beta_T (1 - i_{T_{N,E}} - i_T) \left( \alpha i_M + (1 - \alpha) i_A \right) - \eta_T i_{T_{N,E}} - \mu_T i_{T_{N,E}} \\
\frac{di_T}{dt} &= \eta_T i_{T_L} - \eta_T i_T - \mu_T i_T \\
\frac{di_{T_N}}{dt} &= \eta_T i_{T_N} - \mu_T i_{T_N} \\
\frac{di_{T_A}}{dt} &= \eta_T (i_{T_N} + i_{T_{N,E}}) - \mu_T i_{T_A} \\
\frac{di_A}{dt} &= \beta_A (1 - i_A) \left( i_{T_A} + (1 - \alpha) i_T \right) - \mu_A i_A
\end{align*}
\]
## Parameters

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<td>Birth rate of the alternate host into the susceptible Larvae class</td>
</tr>
<tr>
<td>$\beta_M$</td>
<td>Contact Transmission Rate for the mice</td>
</tr>
<tr>
<td>$\beta_{TL}$</td>
<td>Contact Transmission Rate for the larval tick</td>
</tr>
<tr>
<td>$\beta_{TN}$</td>
<td>Contact Transmission Rate for the nymphal tick</td>
</tr>
<tr>
<td>$\beta_A$</td>
<td>Contact Transmission Rate for the alternate host</td>
</tr>
<tr>
<td>$\eta_{TL}$</td>
<td>Rate that the larvae molt into nymphs</td>
</tr>
<tr>
<td>$\eta_{TN}$</td>
<td>Rate that the larvae nymphs into adults</td>
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**Table:** Table of Variables & Parameters for Model with 3 Tick Classes
Basic Reproduction Number of a Infection

\[ R_0 := \max \left\{ \sqrt{\frac{\alpha \beta_M \beta_L \eta_{T_L}}{\mu_M (\eta_{T_L} + \mu_{T_L}) (\eta_{T_N} + \mu_{T_N})}}, \sqrt{\frac{\beta_{T_N} (1 - \alpha) \beta_A \eta_{T_N}}{(\eta_{T_N} + \mu_{T_N}) \mu_{T_A} \mu_A}} \right\} \]
To illustrate that the disease will be endemic in the alternate host. We chose variables in the following manner in the following manner:

- higher $\beta_M$
- lower $\beta_{TL}$, $\beta_{TN}$, and $\beta_A$
Model with 3 Tick Classes $\alpha = 1, \beta_M = .21, \beta_{TL} = .00041, \beta_{TN} = .00041, \beta_A = .00041, R_0 = 6.2214$
Model with 3 Tick Classes $\alpha = 1, \beta_M = .01, \beta_{TL} = .0041, \beta_{TN} = .0041, \beta_A = .0041, R_0 = 2.9626$
Conclusion

- It’s very beneficial for the mice to be able to sustain the disease, but not necessary.
- If the disease is endemic to the mice population, then it’s very likely that the disease will be endemic for the alternate host population.
System of ODEs Modeling the Spread of Lyme Disease

\[ \frac{di_M}{dt} = \delta_3 \alpha \beta_M (1 - i_M)i_T - \mu_M i_M \] (2a)

\[ \frac{di_{T_L}}{dt} = \delta_1 \beta_{T_L} (1 - i_{T_L})i_M - \eta_T i_{T_L} - \mu_{T_L} i_{T_L} \] (2b)

\[ \frac{di_{T_{N,E}}}{dt} = \delta_3 \beta_T (1 - i_{T_{N,E}} - i_T) \left( \alpha i_M + (1 - \alpha) i_A \right) - \eta_{TN} i_{T_{N,E}} - \mu_T i_{T_{N,E}} \] (2c)

\[ \frac{di_{T_N}}{dt} = \eta_T i_{T_L} - \eta_{TN} i_{T_N} - \mu_T i_{T_N} \] (2d)

\[ \frac{di_{T_A}}{dt} = \eta_T (i_{T_N} + i_{T_{N,E}}) - \mu_A i_{T_A} \] (2e)

\[ \frac{di_A}{dt} = \beta_A (1 - i_A) \left( \delta_5 i_T + \delta_3 (1 - \alpha) i_T \right) - \mu_A i_A \] (2f)
\[
\delta_1 = \begin{cases} 
1 & \text{if active} > 182 \text{ active} < 283 \\
0 & \text{not active}
\end{cases}
\]

\[
\delta_3 = \begin{cases} 
1 & : \text{active} > 119 \text{ and active} < 283 \\
0 & \text{not active}
\end{cases}
\]

\[
\delta_5 = \begin{cases} 
1 & \text{active} > 274 \text{ active} < 346 \text{ or active} > 41 \text{ active} < 161 \\
0 & \text{not active}
\end{cases}
\]
Basic Reproduction Number of an Infection

\[ R_0 := \begin{cases} 
0 & 0 \leq t \leq 121 \\
\sqrt{\frac{\alpha \beta_M \beta_T \eta_T L}{(\eta_T + \mu_T L)(\eta_T N + \eta_T N)\mu_M}} & 121 \leq t \leq 274 \\
\max \left\{ \sqrt{\frac{\alpha \beta_M \beta_T L \eta_T M}{\mu_M(\eta_T + \mu_T L)(\eta_T N + \mu_T N)}}, \sqrt{\frac{\beta_T N (1 - \alpha) \beta_A \eta_T N}{(\eta_T + \mu_T N)\mu_T A \mu_A}} \right\} & 274 \leq t \leq 283 \\
0 & 283 \leq t \leq 346 \\
0 & 346 \leq t \leq 365 
\end{cases} \]
Model with Seasonality: $\alpha = 1, \beta_M = .0011, \beta_{TL} = .0011, \beta_{TN} = .0011, \beta_A = .0011, R_0 = .8693$
Model with Seasonality: $\alpha = 0.5, \beta_M = 0.011, \beta_{TL} = 0.011, \beta_{TN} = 0.011, \beta_A = 0.011, R_0 = 0.7561$
Model with Seasonality: $\alpha = 0, \beta_M = .0011, \beta_{TL} = .0011, \beta_{TN} = .0011, \beta_A = .0011, R_0 = .7036$
Lyme disease is found in mice in Texas at low levels. The model indicates that it’s very likely that there is a larger host that is also an effective carrier.
References
