Refining Fewnomial Theory for $2 \times 2$ Systems

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Descartes’ Rule of Signs (17th century)

If \( f(x) := c_1x^{a_1} + \cdots + c_Tx^{a_T} \in \mathbb{R}[x, x^{-1}] \) and \((a_1 < \cdots < a_t)\), then the number of positive roots (counting multiplicity) is less than or equal to the number of sign alternations in \((c_1, \cdots, c_T)\).

- Direct consequence is that the maximum finite number of positive roots is \((T - 1)\)
- Relating Descartes’ Rule to multivariable systems of polynomials remains a difficult open problem
Definitions

Definition
We define a 2×2 System as a system of two polynomials and two variables.

Definition
We define a systems of two variables where one is a trinomial and the other is an \( m \)-nomial as a System of Type (3,m).

Example:

\[
\beta + x^{r_2} y^{s_2} + x^{r_3} y^{s_3} + \cdots + \alpha_m x^{a_m} y^{b_m}
\]

where \( \beta, \alpha_1, \ldots, \alpha_m \in \mathbb{R} \).
We look at systems of type (3,m):

\[ \beta + x^{r_2}y^{s_2} + x^{r_3}y^{s_3} + \alpha_1 + \alpha_2x^{a_2}y^{b_2} + \cdots + \alpha_mx^{a_m}y^{b_m} \]

where \( \beta, \alpha_1, \ldots, \alpha_m \in \mathbb{R} \).

- The maximum finite number of roots in \( \mathbb{R}^2_+ \) of systems of type (3, m) is known to lie between \( 2m - 1 \) and \( \frac{2}{3}m^3 + 5m \).
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- The maximum finite number of roots in $\mathbb{R}^2_+$ of systems of type $(3,m)$ is known to lie between $2m - 1$ and $\frac{2}{3}m^3 + 5m$.
- We want to tighten current bounds.
- We want to construct new extremal examples of minimal height (simpler examples).
Techniques

Rolle’s Theorem

If \( f : [a, b] \longrightarrow \mathbb{R} \) is continuous and differentiable, and \( f(a) = f(b) \), then there is a \( c \in (a, b) \) such that \( f'(c) = 0 \).

- Techniques applied to this problem have been variants of Rolle’s Theorem and a result of Polya on the Wronskian.
- We will consider intersections of convex arcs.

Figure: Rolle’s Theorem
Techniques

Rolle’s Theorem

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Figure: Rolle’s Theorem
Figure: Haas, 2000
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New systems also lead to derivations of new facts

The idea so to create a system of type \((3, m)\) with \(2m - 1\) roots in \(\mathbb{R}^2_+\) in order to get a system of type \((3, m + 1)\) with \(2(m + 1) - 1\) roots in \(\mathbb{R}^2_+\)

So we will start with a system of type \((3, 3)\) to construct a system of type \((3, 4)\)
In 2000, Haas found the first $2 \times 2$ system of type $(3, 3)$ with 5 roots in $\mathbb{R}_2^2$

\[ y^{106} + 1.1x^{53} - 1.1x \]
\[ x^{106} + 1.1y^{53} - 1.1y \]

In 2007, the simplest $2 \times 2$ system of type $(3,3)$ with 5 roots in $\mathbb{R}_2^2$, discovered by Dickenstein, Rojas, Rosek, and Shih was found:

\[ x^6 + \frac{44}{31}y^3 - y \]
\[ y^6 + \frac{44}{31}x^3 - x \]
We specifically look at the following system:

\[ f(x_1, x_2) := x_1^5 - \frac{49}{95} x_1^3 x_2 + x_2^6 \]
\[ g(x_1, x_2) := x_2^5 - \frac{49}{95} x_1 x_2^3 + x_1^6 \]

- We verify we have 5 roots in \( \mathbb{R}_+^2 \)
- We reduce the system
- Construct a \( 2 \times 2 \) system of type \((3, 4)\) by adding a monomial term
We start with

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\[ g(x_1, x_2) := x_2^5 - \frac{49}{95} x_1 x_2^3 + x_1^6 \]

By rescaling and performing a change of variables, we got

\[ r(u, v) := u - \frac{49}{95} + v \]
\[ s(u, v) := u^{\frac{1}{7}} v^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} v^{-\frac{1}{7}} \]

where \( u = x_1^2 x_2^{-1} \) and \( v = x_1^{-3} x_2^5 \)
Finding Roots

\[ r(u, v) := u - \frac{49}{95} + v \]

\[ s(u, v) := u^{1/7} v^{3/7} - \frac{49}{95} + u^{16/7} v^{-1/7} \]

Setting \( r = s = 0 \), we get the following algebraic function:

\[ G(u) := u^{1/7} \left( \frac{49}{95} - u \right)^{3/7} - \frac{49}{95} + u^{16/7} \left( \frac{49}{95} - u \right)^{-1/7} = 0 \]
Finding Roots

We care about roots that lie in the interval \((0, \frac{49}{95})\). Why?

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Recall

We obtained \(G(u)\) by setting \(r = s = 0\). So

\[
r(u, v) := u - \frac{49}{95} + v = 0 \Rightarrow v = \frac{49}{95} - u
\]

- So the roots of \(G(u)\) that lie in \((0, \frac{49}{95})\) imply that \(v\) is also positive.
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- This implies \(x_1, x_2 > 0\)
Finding Roots

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Why do we care about the roots at all?

- Finding roots will give us Regions of Interest to insert a ”hump” that yields 7 intersections with \( G(u) \)
- How do we find these roots?
Finding Roots

**Definition**

Given any \( d, e \in \mathbb{N} \) and \( f, g \in \mathbb{C}[x] \) with \( \deg(f) \leq d \) and \( \deg(g) \leq e \), the **Sylvester Matrix** of \((f, g)\) of format \((d, e)\) is:

\[
\text{SYL}_{(d,e)}(f, g) = \begin{pmatrix}
    a_0 & a_1 & \cdots & a_d & 0 & \cdots & 0 \\
    0 & a_0 & a_1 & \cdots & a_d & \cdots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & a_0 & a_1 & \cdots & a_d \\
    b_0 & b_1 & \cdots & b_e & 0 & \cdots & 0 \\
    0 & b_0 & b_1 & \cdots & b_e & \cdots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & b_0 & b_1 & \cdots & b_e
\end{pmatrix}
\]

**Figure:** Sylvester Matrix of \((f, g)\) of format \((d, e)\)
Definition

Given any $d, e \in \mathbb{N}$ and $f, g \in \mathbb{C}[x]$ with $\deg(f) \leq d$ and $\deg(g) \leq e$, the **Sylvester Matrix** of $(f, g)$ of format $(d, e)$ is:

Definition

The **Resultant** of $f$ and $g$ (denoted $\text{Res}_{(d,e)}(f, g)$) is the determinant of their Sylvester Matrix.
Finding Roots

- We have

\[
\begin{align*}
r(u, v) &:= u - \frac{49}{95} + v \\
s(u, v) &:= u^{\frac{1}{7}} v^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} v^{\frac{-1}{7}}
\end{align*}
\]

- Set \( u = p^{7} \) and \( v = q^{7} \) and multiply \( z(p, q) \) by \( q \)

\[
\begin{align*}
t(p, q) &:= p^{7} - \frac{49}{95} + q^{7} \\
z(p, q) &:= pq^{4} - \frac{49}{95} q + p^{16}
\end{align*}
\]

- Now we get the resultant of \( t \) and \( z \) with respect to \( q \) to find the roots
The resultant yields the following polynomial

\[ p^{112} - \frac{823543}{857375} p^{56} + \cdots - \frac{33232930569601}{6634204312890625} \]
Finding Roots

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\[ p^{112} - \frac{823543}{857375} p^{56} + \cdots - \frac{33232930569601}{6634204312890625} \]

- Recall we had \( u = p^7 \). So we substitute \( p = u^{\frac{1}{7}} \).

\[ u^{16} - \frac{823543}{857375} u^8 + \cdots - \frac{33232930569601}{6634204312890625} \]

- This is an easier polynomial to compute roots
Humps and Bumps

\[ G(u) := u^{\frac{1}{7}} \left( \frac{49}{95} - u \right)^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} \left( \frac{49}{95} - u \right)^{-\frac{1}{7}} = 0 \]

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Constructing New Systems

\[ G(u) := u^{\frac{1}{7}} \left( \frac{49}{95} - u \right)^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} \left( \frac{49}{95} - u \right)^{-\frac{1}{7}} = 0 \]

- So where do the humps come from?
- We use the monomial term

\[ H(u) := cu^a \left( \frac{49}{95} - u \right)^b \]

- For what \((a, b, c)\) does \(G(u)\) and \(H(u)\) have 7 intersections
Humps and Bumps

\[ H(u) := cu^a \left( \frac{49}{95} - u \right)^b \]

How do we choose \((a, b, c)\)?

- We want to insert a hump in some interval \( (i_1, i_2) \)
- We want the peak to be at the midpoint \( \left( \frac{i_1 + i_2}{2} \right) \)
- We want the inflection points to be at the endpoints \( i_1, i_2 \)
Humps and Bumps

\[ H(u) := cu^a \left( \frac{49}{95} - u \right)^b \]

How do we choose \((a, b, c)\)?

- By taking some derivatives and with some algebra we find that

\[ a = \frac{m^2}{d^2} \left( 1 - \frac{95m}{49} \right) + \frac{95m}{49} \]

where \(m = \frac{i_1 + i_2}{2}\) and \(d = \frac{i_2 - i_1}{2}\)
Humps and Bumps

\[ H(u) := cu^a \left( \frac{49}{95} - u \right)^b \]

How do we choose \((a, b, c)\)?

- We also get

\[ b = \frac{49a}{95m} - a \]

where \(m = \frac{i_1 + i_2}{2}\) and

\[ c = h \cdot \left( \frac{a + b}{49/95} \right)^{a+b} \cdot \frac{1}{a^a b^b} \]

where \(h\) is the desired height of the peak of \(H(u)\)
Constructing New Examples

Once we get a \( H(u) \) that intersects \( G(u) \), what is next?

- We let \( G_2(u) = G(u) - H(u) \)

\[
G_2(u) = u^{\frac{1}{7}} \left( \frac{49}{95} - u \right)^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} \left( \frac{49}{95} - u \right)^{-\frac{1}{7}} - cu^a \left( \frac{49}{95} - u \right)^b
\]

**Figure:** \( G(u) \) is red; \( H(u) \) is blue
Constructing New Examples

Once we get a $H(u)$ that intersects $G(u)$, what is next?

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Figure: $G_2(u)$
How do we create a new system?

\[ G_2(u) = u^{\frac{1}{7}} \left( \frac{49}{95} - u \right)^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} \left( \frac{49}{95} - u \right)^{-\frac{1}{7}} - cu^a \left( \frac{49}{95} - u \right)^b \]

- We undo the substitution to get a new system

\[ r_2(u, v) := u - \frac{49}{95} + v \]

\[ s_2(u, v) := u^{\frac{1}{7}} v^{\frac{3}{7}} - \frac{49}{95} + u^{\frac{16}{7}} v^{-\frac{1}{7}} - cu^a v^b \]

- Undo change of variables to get a 2 × 2 system of type (3, 4) with 7 roots in \( \mathbb{R}^2_+ \)
Finding More examples

- We started off by finding two humps for regions 2-5
  - One centered between two endpoints of the region
  - One centered on actual peak of that region
- We then found examples in Regions 1 & 6
- We found examples with humps closer to endpoints

**Figure:** Regions of Interest
Finding More examples

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**Figure: Regions of Interest**

**Figure: Results**
Finding More examples

Please note...

- Scalar multiples work too!
- Colored areas yield more possible examples

Figure: Regions of Interest

Figure: Results
Recall that one of our goals is to find extremal examples of minimal height.
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In 2007, the first known $2 \times 2$ system of type $(3, 4)$ with 7 roots in $\mathbb{R}^2_+$ was discovered by Gomez, Niles, and Rojas:

\begin{align*}
  x^6 + \frac{44}{31} y^3 - y \\
  y^{14} + \frac{44}{31} x^3 y^8 - xy^8 + 1936254x^{133}
\end{align*}
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$$x^6 + \frac{44}{31} y^3 - y$$

$$y^{14} + \frac{44}{31} x^3 y^8 - xy^8 + 1936254x^{133}$$

GOAL! We found a new example!
In 2007, the first known $2 \times 2$ system of type $(3, 4)$ with 7 roots in $\mathbb{R}^2_+$ was discovered by Gomez, Niles, and Rojas.

\[
x^6 + \frac{44}{31} y^3 - y
\]
\[
y^{14} + \frac{44}{31} x^3 y^8 - xy^8 + 1936254x^{133}
\]

My example

\[
x^5 - \frac{49}{95} x^3 y + y^6
\]
\[
x^{33} y^5 - \frac{49}{95} y^3 x^{34} + x^{39} + 5807 y^{62}
\]
Quest to a Simple Example

\[ x^5 - \frac{49}{95} x^3 y + y^6 \]

\[ x^{33} y^5 - \frac{49}{95} y^3 x^{34} + x^{39} + 5807 y^{62} \]

**Figure:** Regions of Interest

**Figure:** Results

Mark Stahl

Refining Fewnomial Theory for 2 × 2 Systems
There are more systems to look at!

\[ 1 + x^4 - \frac{10}{17}x^5 y^2 \]

\[ 1 + y^4 - \frac{10}{17}x^2 y^5 \]
New Direction

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\[ 1 + x^4 - \frac{10}{17}x^5y^2 \]
\[ 1 + y^4 - \frac{10}{17}x^2y^5 \]

Preliminary Result:

\[ 1 + x^4 - \frac{10}{17}x^5y^2 \]
\[ 1 + y^4 - \frac{10}{17}x^2y^5 + 102000x^{-94}y^{-35} \]
We now look to prove the following:

**Conjecture**

Let us fix an integer $k$. Then the maximum number of roots of

$$\sum_{i=1}^{k} c_i u^{a_i} (1 - u)^{b_i} \text{ in } (0, 1)$$

(over all real $a_i$, $b_i$, and $c_i$) is $O(k)$. 