Zeros of the Modular Form $E_k E_l - E_{k+l}$

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July 23, 2015
A complex-valued function $f$ is a modular form if:

1. for all $z \in \mathbb{H} \cup \{\infty\}$, $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$ we have $f(\gamma(z)) = f(az + bcz + d) = (cz + d)^k f(z)$. Then $k$ is called the weight of $f$. (Transformation Law)

2. $f$ is holomorphic (complex differentiable) for every point $z \in \mathbb{H} \cup \{\infty\}$, so $f$ can be expanded in a power series in $z$ around any point $z_0 \in \mathbb{H}$.

We look at modular forms of positive, even weight. The valence formula tells us how many zeros $f$ has. $k = \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum_{z \neq i, \rho} v_z(f)$. 

Sarah Reitzes and Polina Vulakh (Tufts University, Bard College) 
Eisenstein Series 
July 23, 2015
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1. For all $z \in \mathbb{H} \cup \{\infty\}$, if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$, we have $f(\gamma(z)) = f(az + b/cz + d) = (cz + d)^k f(z)$. Then $k$ is called the **weight** of $f$. (Transformation Law)

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We look at modular forms of positive, even weight. The **valence formula** tells us how many zeros $f$ has:

$$k/12 = \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \sum_{z \neq i, \rho} v_z(f)$$

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Eisenstein Series

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A complex-valued function $f$ is a **modular form** if:

- for all $z \in \mathbb{H} \cup \{\infty\}$, $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$

we have $f(\gamma(z)) = f(\frac{az+b}{cz+d}) = (cz + d)^k f(z)$. Then $k$ is called the weight of $f$. (*Transformation Law*)
A complex-valued function $f$ is a **modular form** if:

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Zeros of Modular Forms

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We look at modular forms of positive, even weight.

The **valence formula** tells us how many zeros $f$ has.

$$\frac{k}{12} = \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \sum_{z \neq i, \rho, z \in \mathbb{H}} v_z(f)$$
Eisenstein series of weight $k$:

$$E_k(z) = \frac{1}{2} \sum_{(c,d)=1 \atop c,d \in \mathbb{Z}} \frac{1}{(cz+d)^k} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}$$

It has been proven that the zeros of the Eisenstein series lie on the arc of the fundamental domain $\mathcal{F} = \{ z = x + iy \in \mathbb{H} : x \in (-\frac{1}{2}, \frac{1}{2}), |z| \geq 1 \}$ (RSD, 1970).

Figure: $\mathcal{F}$  
Figure: Zeros of $E_{70}$
Zeros of $E_k E_l - E_{k+l}$

Conjecture:

The zeros of $E_k E_l - E_{k+l}$, a modular form of weight $k + l$, lie on the boundary of $\mathcal{F}$.

Figure: Zeros of $E_{50}^2 - E_{100}$

Conjecture:

The zeros of $E_k^2 - E_{2k}$, a modular form of weight $2k$, lie on the lines $x = \pm \frac{1}{2}$ in $\mathcal{F}$.
Proving the zeros of $E_k^2 - E_{2k}$

Since $E_k(\frac{1}{2} + iy)$ is real-valued, we prove the desired number of zeros \(\lfloor \frac{k}{6} \rfloor - (1 + n)\) via IVT using points of the form $\frac{1}{2} + iy_m$ where $y_m = \frac{\tan(\theta_m)}{2}$ for $\theta_m = \frac{m\pi}{k}$ where $m \in \mathbb{Z}$ such that $\lceil \frac{k}{3} \rceil \leq m < \frac{k}{2} - n$.

Why $-n$?

We run into problems for $y \geq \frac{c_0 \sqrt{k}}{\sqrt{\log k}}$, so $n$ is the number of zeros with $y$ past this range.

However, there exists a method involving the Fourier expansion that proves the location of zeros for which $y > c_1 \sqrt{k \log k}$, so we lose very few zeros altogether.
Approximating $E_k^2 - E_{2k}$

Write $E_k(\frac{1}{2} + iy) = M_k(\frac{1}{2} + iy) + R_k(\frac{1}{2} + iy)$ where $M_k$ corresponds to $c^2 + d^2 \leq 1$ - except for $(c, d) = (1, 1)$ - and $R_k$ corresponds to all other $(c, d)$.

Then

$$E_k^2(\frac{1}{2} + iy) - E_{2k}(\frac{1}{2} + iy) = (M_k(\frac{1}{2} + iy) + R_k(\frac{1}{2} + iy))^2$$

$$- (M_{2k}(\frac{1}{2} + iy) + R_{2k}(\frac{1}{2} + iy))$$

$$= M_k(\frac{1}{2} + iy)^2 + 2M_k(\frac{1}{2} + iy)R_k(\frac{1}{2} + iy)$$

$$+ R_k(\frac{1}{2} + iy)^2 - M_{2k}(\frac{1}{2} + iy) - R_{2k}(\frac{1}{2} + iy)$$

We know $|R_k(\frac{1}{2} + iy)| < \frac{9+12y}{\left(\frac{9}{4}+y^2\right)^k}$, which is decreasing in $k$, and since

$M_k(\frac{1}{2} + iy) = 1 + \frac{1}{(\frac{1}{2}+iy)^k} + \frac{1}{(-\frac{1}{2}+iy)^k}$, we know $|M_k(\frac{1}{2} + iy)| \leq 3$. Then we want to show

$$|M_k(\frac{1}{2} + iy_m)^2 - M_{2k}(\frac{1}{2} + iy_m)| > 8\left(\frac{9+12y_m}{\left(\frac{9}{4}+y_m^2\right)^k}\right)$$
Approximating $E_k^2 - E_{2k}$ (cont.)

For our points $\frac{1}{2} + iy_m$, we have a lower bound

$$|M_k(\frac{1}{2} + iy_m)^2 - M_{2k}(\frac{1}{2} + iy_m)| \geq \frac{4(\frac{1}{4} + y_m^2)^k}{(\frac{1}{4} + y_m^2)^k} - 2$$

so we want to show

$$\frac{4(\frac{1}{4} + y_m^2)^k}{(\frac{1}{4} + y_m^2)^k} - 2 > 8\left(\frac{9 + 12y_m}{\left(\frac{9}{4} + y_m^2\right)^{\frac{k}{2}}}\right)$$

For large $y$, this is not true: specifically for $y \geq c_0 \frac{\sqrt{k}}{\sqrt{\log k}}$ where $c_0 \leq \frac{1}{\sqrt{8}}$, so we work with $y_m < c_0 \frac{\sqrt{k}}{\sqrt{\log k}}$.

By simplifying further, we have $\left(\frac{\frac{9}{4} + y_m^2}{\frac{9}{4} + y_m^2}\right)^{\frac{k}{2}} > c_2 y_m$ where $c_2 = \frac{38}{\sqrt{3}} + 24$.

This is true for $k \geq c_2$, so we have proved

$$|M_k(\frac{1}{2} + iy_m)^2 - M_{2k}(\frac{1}{2} + iy_m)| > 8\left(\frac{9 + 12y_m}{\left(\frac{9}{4} + y_m^2\right)^{\frac{k}{2}}}\right).$$
Sign changes from $M_k\left(\frac{1}{2} + iy_m\right)^2 - M_{2k}\left(\frac{1}{2} + iy_m\right)$

If we rewrite $\frac{1}{2} + iy_m = re^{i\theta_m}$, we have

$$M_k(\text{re}^{i\theta_m})^2 - M_{2k}(\text{re}^{i\theta_m}) = \frac{4r^k(-1)^m + 2}{r^{2k}}$$

For $\theta_m = \frac{m\pi}{k}$ where $m \in \mathbb{Z}$ such that $\left\lfloor \frac{k}{3} \right\rfloor \leq m < \frac{k}{2} - n$, this yields $\left\lfloor \frac{k}{6} \right\rfloor - n$ sign changes corresponding to $\left\lfloor \frac{k}{6} \right\rfloor - n - 1$ zeros by IVT.
Extending this to general $E_kE_l - E_{k+l}$

Recall that $B_{k,l} =$ number of zeros of $E_kE_l - E_{k+l}$ for which $x = \frac{1}{2}$.

**Conjecture:**

$(k \geq l)$ The number of zeros $E_kE_l - E_{k+l}$ for which $x = \frac{1}{2}$ is at least that of $E_l^2 - E_{2l}$. In other words, $B_{k,l} \geq B_{l,l}$. 

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Example: $E_k E_{34} - E_{k+34}$

Figure: $k=34$

Figure: $k=40$

Figure: $k=44$

Figure: $k=50$
Extending this to general $E_kE_l - E_{k+l}$ (cont.)

Our main term becomes

$$M_k(re^{i\theta})M_l(re^{i\theta}) - M_{k+l}(re^{i\theta}) = \frac{r^{2l+k+2}\cos(\theta k) + r^{2k+l+2}\cos(\theta l) + r^{k+l+2}\cos(\theta(k-l))}{r^{2(k+l)}}$$

If we rewrite $k = l + d$ and let $\theta_m = \frac{m\pi}{l}$ for $\lfloor \frac{l}{3} \rfloor \leq m < \frac{l}{2}$,

$$\frac{r^{3l+d}2(-1)^m \cos\left(\frac{m\pi}{l}d\right) + r^{3l+2d}2(-1)^m + r^{2l+d}2\cos\left(\frac{m\pi}{l}d\right)}{r^{4l+2d}}$$

as our main term instead.

By splitting this up into three cases for $d \equiv 0, 2, 4 \pmod{6}$, we follow a similar method to show that $E_kE_l - E_{k+l}$ has at least $\lfloor \frac{l}{6} \rfloor - n - 1$ zeros or which $x = \frac{1}{2}$.