Counting and Finding Real Roots of Univariate Trinomials

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Introduction

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where $c \neq 0$, $0 < m < n$, and $gcd(m, n) = 1$ and defining

$$r_{m,n} := \left| \frac{n}{m^{\frac{m}{n}}(n-m)^{\frac{(n-m)}{n}}} \right|$$

there are various cases when we compare the coefficient c to $r_{m,n}$.

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- The larger root of f is given by

$$x_{hi}(c) = c^{\frac{1}{(n-m)}}\left[1 - \sum_{k=1}^{\infty}\left(\frac{1}{k(n-m)^k}\cdot\prod_{j=1}^{k-1}\frac{km+j(n-m)-1}{j}\right)\frac{1}{c^{\frac{kn}{(n-m)}}}\right]$$

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However, how can you solve without knowing how many roots there are?

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By examining the sign of the coefficients, we can determine the number of roots f will have

1. $c_1, c_2 > 0 \Rightarrow 0$ roots
2. $c_1, c_2 < 0 \Rightarrow 1$ root
3. $c_1 < 0$ and $c_2 > 0 \Rightarrow 1$ root

Root Counting

When $c_1 > 0$ and $c_2 < 0$, f can have 0, 1, or 2 roots.
By evaluating the modified A-discriminant:

$$\Xi_A = \left(\frac{c_1}{a_3 - a_2}\right)^{a_3 - a_2} \left(\frac{c_2}{-a_3}\right)^{-a_3} \left(\frac{c_3}{a_2}\right)^{a_2} - 1$$

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if this value is:

1. $> 0 \Rightarrow 2$ roots
2. $= 0 \Rightarrow 1$ root
3. $< 0 \Rightarrow 0$ roots

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Baker's Theorem (1966)

If $\alpha_i \in \mathbb{Q}_+$, $b_i \in \mathbb{Z}$ with $logA_i := max\{h(\alpha_i), |log(\alpha_i)|, 0.16\}$, $B := max\{|b_i|\}$, then

$$\sum_{i=i}^{m} b_i log(\alpha_i) \neq 0 \Rightarrow$$

$$log|\sum_{i=i}^{m} b_i log(\alpha_i)| > -1.4 \cdot m^{4.5} 30^{m+3}(1 + log(B)) \prod_{i=1}^{m} log(A_i)$$

Approximating Logarithms

Lemma

Given any $x \in \mathbb{Q}_+$ of height h, and $\ell \in \mathbb{N}$ with $\ell \geq h$ we can compute $\lfloor log_2 max\{1, log(x)\}\rfloor$ and the ℓ most significant bits of $log(x)$ in time $O(\ell log^2(\ell))$.

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We can approximate using

$$11L_1 - 13L_2 + 2L_3$$

where $L_i \approx log(\alpha_i)$ up to error $<$
$\frac{1}{10}(-1.4 \cdot m^{4.5}30^{m+3}(1 + log(B))\prod_{i=1}^{m} log(A_i))$

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So, we need a new series to solve this case.

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$$x^{\pm}_{sing} = \zeta \sum_{k=0}^{\infty} \frac{\gamma_k}{(\pm\sqrt{(n-m)r})^k}(c-r)^{k/2}$$

where $\zeta = \left(\frac{m}{n-m}\right)^{1/n}$ and $\gamma_k \in \mathbb{Q}[m, n]$

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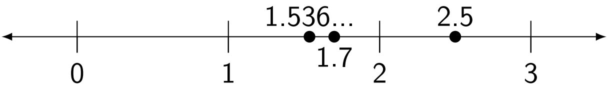
We know γ_k has degree $k-1$ in (m, n), but an explicit formula is not currently known.

Comparing Series



Terms needed for error $< 1/1000$ for approximation of larger root of $f(x) = 1 - cx^2 + x^{13}$ for varying values of c:

c	x_{hi}	x^{+}_{sing}
1.5362173	$>$20000	2
1.7	17	5
2.5	5	9

Thank you!