Developing a New Tool for Modeling the Topology of Zero Sets of Bivariate Pentanomials

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Overview

- Terminology and Background
- Motivation and Goals
- Matlab Program
- Results

Near Circuit Polynomials

Support

Def: Given a polynomial f, the **support** is its set of exponent vectors

E.g. $f(x, y) = 1 - x - y + x^4 y + xy^4$, support $\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$

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Near Circuit Polynomials

Def: A polynomial whose support $\mathcal{A} = [a_1, \ldots, a_{n+3}] \in \mathbb{Z}^{n\times(n+3)}$ yields $\begin{bmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{n+3} \end{bmatrix}$ having rank $n+1$.

E.g. a **bivariate pentanomial** has 2 variables and 5 terms.

$n =$ the number of variables

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$\mathcal{A}$-**discriminant variety**: where $\mathcal{A}$-discriminant $= 0$
i.e. critical points/curves where the topology of the zero set changes

Reduced[1] $\mathcal{A}$-discriminant variety for $\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$

$$
\begin{aligned}
&363087263602825104457728a^{32}b^{8} - 2904698108822600835661824a^{29}b^{11} + \\
&10166443380879102924816384a^{26}b^{14} - 20332886761758205849632768a^{23}b^{17} + \\
&25416108452197757312040960a^{20}b^{20} - 20332886761758205849632768a^{17}b^{23} + \\
&10166443380879102924816384a^{14}b^{26} - 2904698108822600835661824a^{11}b^{29} + \\
&363087263602825104457728a^{8}b^{32} - 726174527205650208915456a^{31}b^{4} + 5798049740657613386809344a^{28}b^{7} + \\
&3128223705478090040593 6128a^{25}b^{10} - 50571247933680984080252928a^{22}b^{13} - \\
&19129025553375088862665 1136a^{19}b^{16} + 48223661844948968014243 4304a^{16}b^{19} - \\
&36318938189571339901835 6736a^{13}b^{22} + 7448962142351708783640 5760a^{10}b^{25} + \\
&1269670774911129037150 6176a^{7}b^{28} - 726174527205650208915456a^{4}b^{31} + 363087263602825104457728a^{30} - \\
&2893351631835012551147520a^{27}b^{3} - 9297323772275431783268 3520a^{24}b^{6} + \\
&13470366556574773615263 7440a^{21}b^{9} + 25351194225538809508921 34400a^{18}b^{12} + \\
&6930726608820725492905672704a^{15}b^{15} + 10397247952186084766590697472a^{12}b^{18} + \\
&1368264254117216589547831296a^{9}b^{21} + 1788103497072364267461 67296a^{6}b^{24} - \\
&97920096402886895358443 52a^{3}b^{27} + 363087263602825104457728b^{30} + \\
&51524645931445780035403776a^{23}b^{2} - 382889518651229479821639 68a^{20}b^{5} - \\
&459434896114086755201292 6976a^{17}b^{8} - 1813816331637440665952767 1808a^{14}b^{11} - \\
&2131928212143098218696356 6592a^{11}b^{14} + 2514558123743644571580497920a^{8}b^{17} - \\
&269737322421295126029533184a^{5}b^{20} - 20941053496075364622925824a^{2}b^{23} - \\
&2551128356732845719499571 2a^{19}b + 2225676679631729339955937280a^{16}b^{4} + \\
&1359100006303368527105490 9440a^{13}b^{7} + 14323107664774924348979937280a^{10}b^{10} - \\
&11483443502644561069909999616a^{7}b^{13} + 384254443547034707078152192a^{4}b^{16} + \\
&33392996500536631555522560ab^{19} - 5579692232310799015608 32a^{15} - \\
&2845499698372999866809843712a^{12}b^{3} - 4692084142913135619868721152a^{9}b^{6} + \\
&8896118143687124537286066176a^{6}b^{9} - 82843494158262383800850 8416a^{3}b^{12} - \\
&557969223231079901560832b^{15} + 16445468110480590903666278 40a^{8}b^{2} - \\
&9711410059602431138149171 20a^{5}b^{5} + 4910693845839500651939758 08a^{2}b^{8} - \\
&3945942476683996789577871 36a^{4}b - 85689226175777908279603 20ab^{4} + 41987654504771523593992227
\end{aligned}
$$

[1] Reduced coefficient vector is $c := [1, 1, 1, a, b]$

Modeling the $\mathcal{A}$-discriminant Variety

Parametrize the $\mathcal{A}$-discriminant variety: **Horn-Kapranov Uniformization**

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2. let $\mathcal{B}$ be a 2×5 matrix whose columns form a basis for the right nullspace of $\hat{\mathcal{A}}$
3. let λ be the variable of parametrization
4. $\log |\lambda \cdot \mathcal{B}^\top| \cdot \mathcal{B}$ parametrizes the $\mathcal{A}$-discriminant variety

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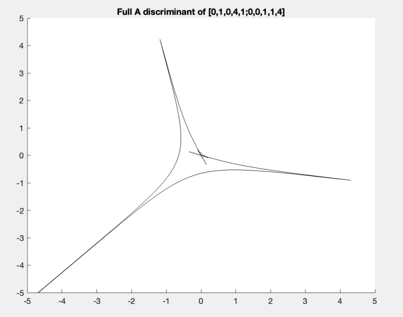
Parametrized $\mathcal{A}$-discriminant Variety



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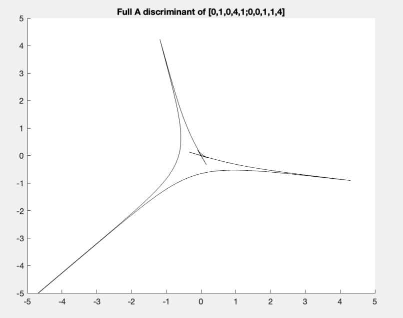
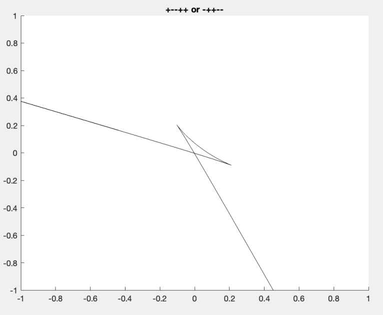


Signed Contour: $+--++$



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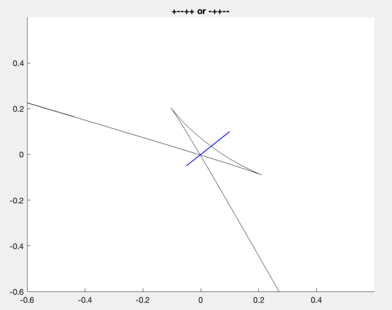

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coefficients:

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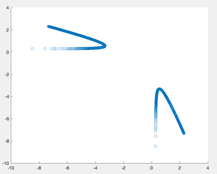
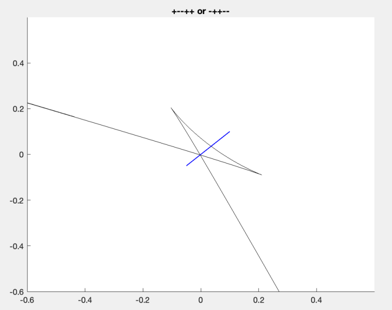
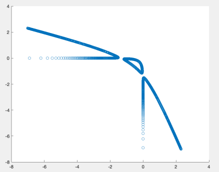
$[1, -\frac{3}{4}, -\frac{3}{4}, 1, 1]$

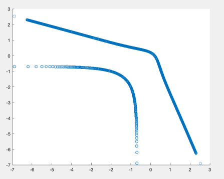




$[1, -1, -1, 1, 1]$



$[\frac{1}{2}, -1, -1, \frac{1}{2}, \frac{1}{2}]$



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- Continue developing approximations to determine which chamber a given coefficient vector lies in.

Program: Detecting Cusps and Inner Chambers

Rusekshih 2017

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- For $n = 2 \rightarrow$ at most 2 cusps within one signed contour.
- If one signed contour has two cusps, we may have an inner chamber.
- To eliminate the possibility of having an inner chamber, detect signed contours with two cusps (cusp: $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$).

Background on Triangulations and Viro's Patchworking

Simple e.g. $f(x) = c_1 x + c_2 y - c_3 xy$: $\mathcal{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

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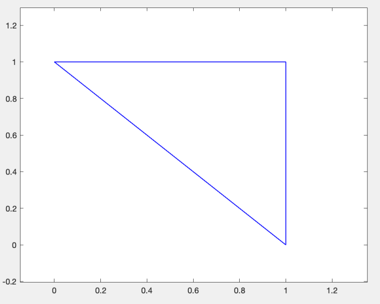

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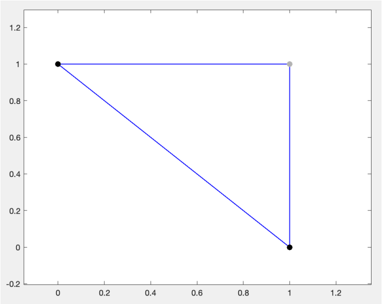

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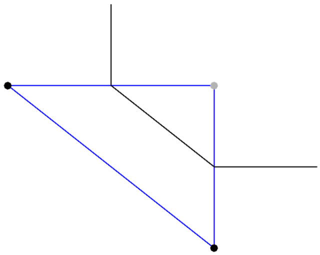

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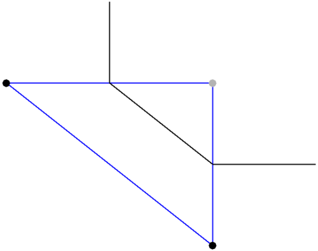

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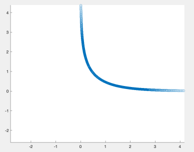

actual zero set

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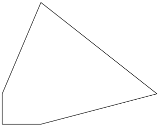

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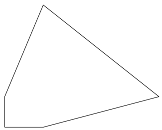

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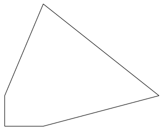

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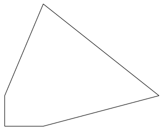

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- Add $-\log|c|$ as third row to support $\mathcal{A}$, where $c = [c_1, c_2, c_3, c_4, c_5]$
- Compute convex hull of lifted support

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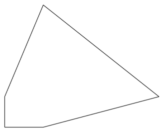

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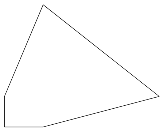

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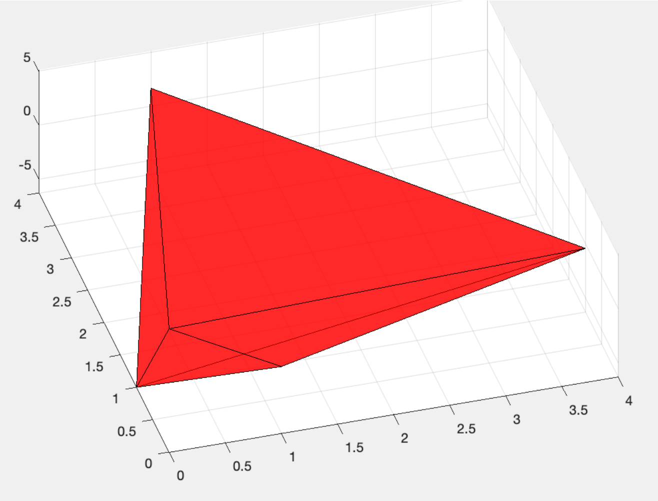
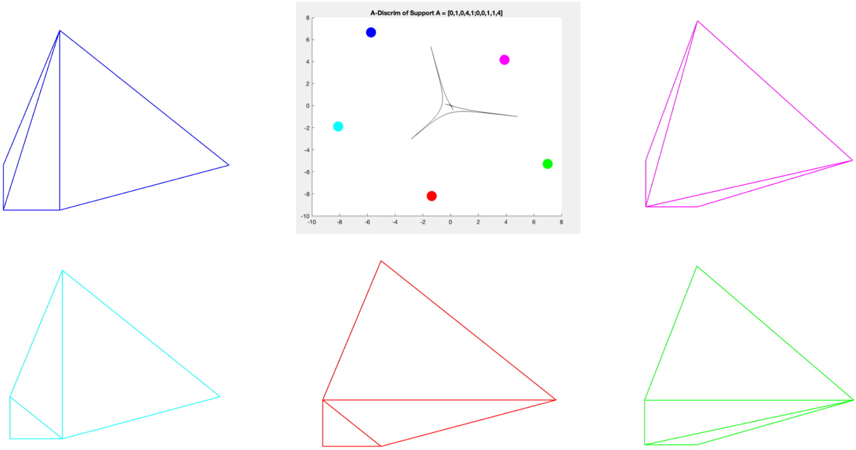
- These triangles form triangulation

Lifted Triangulation Example



Triangulations of $f(x) = c_1 + c_2x + c_3y + c_4x^4y + c_5xy^4$



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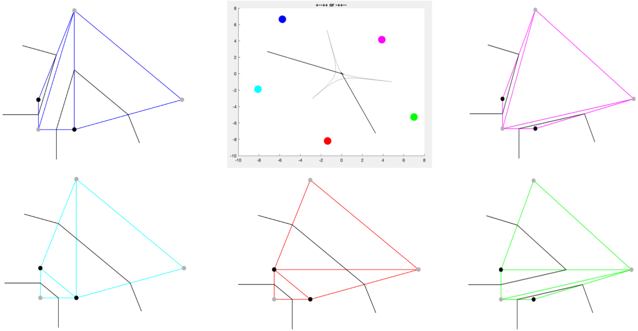

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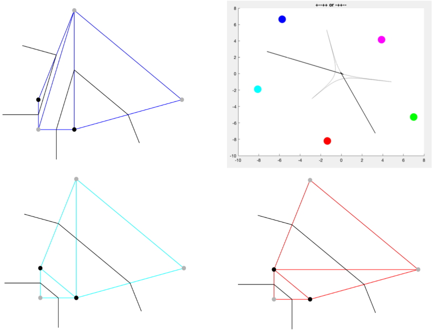
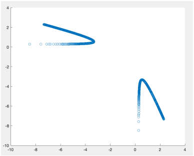


Recall Actual Zero Sets



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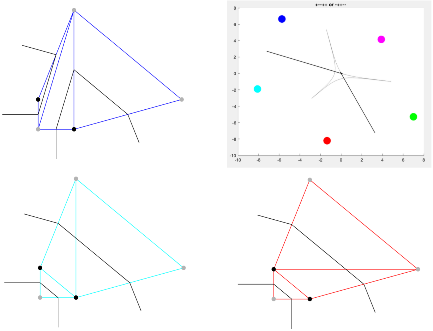
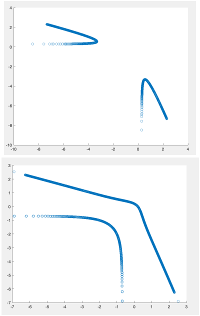


Recall Actual Zero Sets



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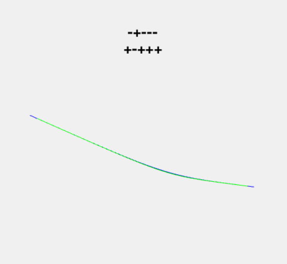

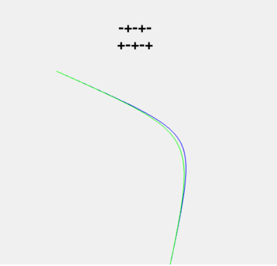

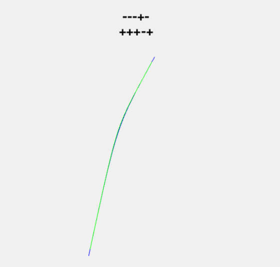

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1. Use two of Ellen's approximations: ray 1 to cusp, cusp to ray 2
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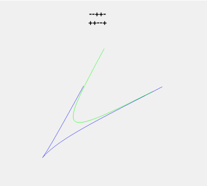
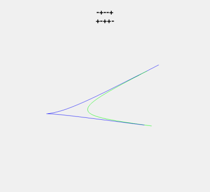
   - $\mathcal{A}$-discriminant of cubic (Support $= [0, 1, 2, 3]$) has a cusp
   - Solving for sidedness of the cubic A-discriminant (not parametrization) is better

Results: Approximating Signed Contours with Cusps

Ellen's Approximations





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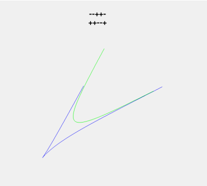

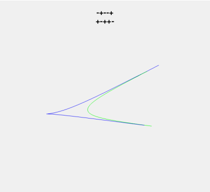

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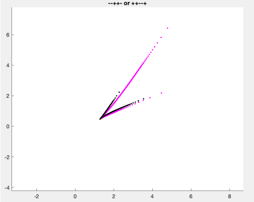
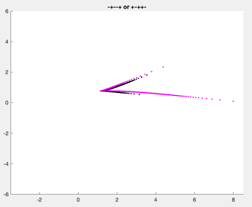
Two of Ellen's Approximations





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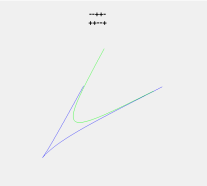

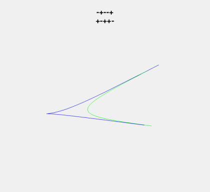

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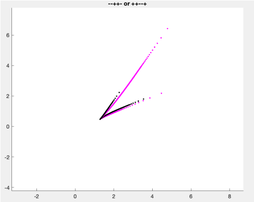
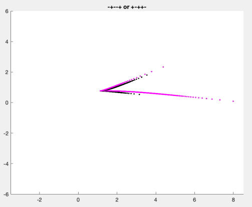
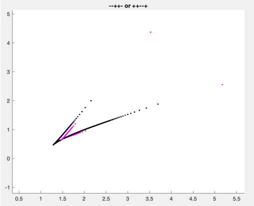
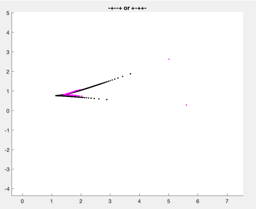




Mapping Cubic Cusp





Thank you!